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Synchronization Phenomena with Nonlinear Oscillators

Dissertação

José Manuel Macedo Cota Barros

Mestrado em Física
(Área de Especialização Física Estatística e Não Linear)
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Orientação: Professora Doutora Ana Maria Ribeiro Ferreira Nunes

2013

$\sim$ *To The Memory Of My Father* $\sim$

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Nomenclature

$\mathcal{R}$	Real set
KBAM	Krylov Bogoliubov Averaging Method - sometimes we only write KB
NN	Nearest neighbours
Sync	Synchronization, or relative to
t	Time - the time variable t is, when possible, omitted
vdP	van der Pol
$\mathcal{I}$	Imaginary set
$\dot{x}$	The time derivative of x
ϕ,φ	Phase
A,$\mathcal{A}$	Amplitude

Preface

The energy of a single thought may determine the motion of a universe.

NIKOLA TESLA

Nature is an infinite network of interacting objects. The idea of interacting itself is redundant - interaction is what we can observe, but in fact we are watching the result of nature's dynamical motion as a whole, in a particular domain. In this work we discuss synchronization, a state where different units in nature evolve together. The goal is to provide information about this subject, and it is directed to anyone. The language herein is truly simple, and every complicated detail is explained. In the margins you will find names of files contained in the multimedia appendix - these files were programmed by me in *Mathematica*. Sync is linked to topics in mathematics, physics, chemistry, biology, psychology, sociology, and many more, so anyone who wants to know more will enjoy this work.

The opportunity to do this work came up before the end of my bachelors degree, when I had to choose a topic to do a sort of short bachelor's thesis. This topic and its vastness intrigued me so much that I decided to do it as my master's thesis. It has been two and a half years of blood, sweat and tears, , pardon the *cliché.* There were some good and bad moments, and the truth is that in science the word "easy" is rare, being much more frequent words like "hard" or "impossible". That emphasizes the beauty of nature. Sometimes, after weeks of trying to understand small issues, the answer would somewhat magically emerge in my head, like it was inevitable to *look* that way and figure out that all previous thoughts were too manufactured to the natural complexity. Understanding, or trying to understand, the dynamics of life is astonishing. Together with my supervisor, we tried to deliver a simple, easy to read, work, based on our dedication to clarify this topic. Small organized chapters will lead the reader from the first word of Introduction, to the last word of the Conclusion, with the help of guided appendices at the end.

This very work would not be possible without the help and guidance of my supervisor Ana Nunes, who built the idea from scratch. Even with tremendous troubles found during the project, was always present to dodge

them and be the mastermind of the project. Her determination and ambition are a reflection of her amazing background and curriculum, which led her to a place only a few can attain, conquered by the hard workers.

I am very thankful to all Professors who, somehow, helped me in my academic or personal life, especially to Guiomar Evans who helped me with the experimental parts.

I want to thank Marta for the review on the thesis, without which it would look a lot worst. Also her love and support were essential to progress and finish my work, and I am thankful for that.

My colleagues know what I have been doing for the past few years, and most of them participated directly in the quest for answering the complex questions this work delivered. Some of the questions we found in our work originated long and healthy arguments between me and my colleagues, which consequently led to a lot of very nice afternoons. Their help and support were essential to keep my mental health sane. To all of them, and their critical thinking, my deepest thank you.

To all my family who always supported me in order to succeed in the academic life, and believed I could become someone they could be proud of someday. Hopefully I can deliver that feeling from now on.

My dearest friends and their unstoppable love. All the moments that would make the problems seem small or just fade away for a while, were definitely due to you. Without you this work would be impossible, because no person can live without his chosen family. I thank you for your love.

I want to thank *Monte Abraão*, a place where people learn the hard life, where connections are more important than money, and where you have to grow up really quick. Without my childhood, I would probably had become someone incapable of facing such a hard goal. It helped me realise, really soon, that what matters is that you do not give up on your dreams. If you keep working for them, you will win.

One very important acknowledgement has to be done to the reader. The reason why people learn and science is made, is to develop common knowledge by delivering it to the world. So, I want to thank the reader, for caring enough to learn and spread information, and making this world a little better and more informed.

Finally, I want to thank nature itself, for being the most interesting, complex, innocent, pure, amazing, and magnificent thing of all. All my love and hard work is fruit of your beautiful mysteries.

José Barros
August 2013, Lisboa, Portugal

Abstract

Synchronization is a state in nature in which a dynamical system of multiple objects evolve together in a coupled motion. The most fundamental case in mathematics is the case of two coupled self-sustained oscillators, which is the starting point of this work. Coupled nonlinear oscillators is a subject related to many areas of science, from biology to economy. In this study we emphasize nearest neighbours coupling of systems of van der Pol oscillators, with an arbitrary coupling we refer in the text. In chapter 1 we introduce some definitions that are important to the understanding of the basic ideas behind synchronization. Some notions are related to topology and graph theory. Next, we develop a method to analyse coupled nonlinear differential equations, the Krylov-Bogoliubov method. We end chapter 1 discussing the importance of an equation for the phase difference, which should result in a constant value after transient into a synchronous regime. The mathematical tools showed in chapter 1 are used in chapter 2 in a analytical (polar coordinates transformation) and numerical (Krylov-Bogoliubov) analysis of two coupled van der Pol oscillators. The experimental part comes in chapter 3, where we build a circuit model which reproduces the coupled equations. The idea is to analyse, experimentally, the synchronization of oscillators, with a mathematical basis. In chapter 4 we end the analysis of two oscillators with a résumé of the theoretical and experimental results, with plots and conclusions. In chapter 5 we repeat the whole study for three coupled oscillators, in order to simulate the coupling terms for this case, and see the differences from the two oscillators case. This will result in the basis for the n oscillators case we deal in chapter 6, where we analyse ten coupled oscillators and its stability to changes in frequency range, and the order of the units in the ensemble.

Keywords: Synchronization, van der Pol, Differential Equations, Non Linear, Coupling, Circuits

Resumo

A sincronização é um fenómeno sobre organização, ou estados organizados, e que pode ser observado na natureza. Neste trabalho foi imposto o objetivo de estudar e observar este fenómeno, usando ferramentas matemáticas e laboratoriais. Como base para o trabalho foi escolhido o caso de dois osciladores acoplados. Este é o estado fundamental da dinâmica coletiva de sistemas acoplados, ou seja, mesmo no estudo de vários osciladores acoplados ($n > 2$), a dinâmica fundamental é a de primeiros vizinhos. Sabendo as condições necessárias, nomeadamente, a necessidade de um oscilador autossustentado, seguiu-se o modelo do oscilador de van der Pol, que é representado por uma equação diferencial não linear e homogénea.

A dinâmica de osciladores acoplados é um estudo atualmente bastante desenvolvido, o que o torna muito diversificado. Sabendo que um modelo matemático – um oscilador autossustentado – pode modelar o batimento cardíaco, as reações químicas, os pêndulos, e muitos outros casos, o problema inicial é tentar compreender numa expressão geral o acoplamento. Sem levar o estudo da estabilidade à exaustão, referimos que para sistemas da forma do oscilador de van der Pol, com um acoplamento arbitrário que dependa das posições e velocidades, o sistema apresenta regimes estáveis. Desta forma, achámos interessante estudar, em particular, um acoplamento de primeiros vizinhos, através de uma constante denominada de constante de acoplamento.

No capítulo 1 é apresentada uma introdução ao tópico de sincronização, ao oscilador de van der Pol, e à função de acoplamento. São utilizadas noções de topologia, de teoria de grafos, e dos sistemas dinâmicos na construção de uma base sólida para escrever as equações acopladas, justificando os termos de acoplamento. Neste mesmo capítulo são introduzida outras noções importantes na compreensão do fenómeno e dos seus resultados, tal como fase, *detuning*, matriz de Kirchhoff e matriz de output. Como queremos saber resultados para as equações, foi igualmente necessária uma pesquisa de métodos perturbativos para resolução de equações diferenciais não lineares acopladas, a qual resultou no método desenvolvido por Nikolay Krylov e Nikolay Bogoliubov. Este é um método que mostra que, para certas condições, podemos considerar a fase e a amplitude constantes durante um período da oscilação, e como tal, podemos tomar a média temporal. Este método é

desenvolvido no final do capítulo 1. É também justificada a necessidade de, para dois osciladores, encontrarmos uma equação da evolução da sua diferença de fase, o que nos dará uma medida para determinar se ocorreu, ou não, sincronização, visto que o sucesso implica que a diferença de fase, após transiente, deverá ser constante. Para o caso de mais osciladores, trataremos de estudar a diferença de fase entre cada 2 vizinhos.

No capítulo 2 tratamos a sincronização em sua plenitude, voltando a apresentar as equações e resolvendo-as analiticamente e usando o método aproximado de Krylov-Bogoliubov de forma a se obter a expressão para a diferença de fase, mostrando passo-a-passo a sua dedução. Depois voltamos a resolver as equações mas desta feita numericamente, utilizando uma mudança de variáveis para coordenadas polares e obtendo expressões explícitas para as fases e as amplitudes de cada oscilador, e integrando a diferença de fase entre eles. Neste capítulo o objetivo passa por derivar estas equações, sendo que os resultados são apresentados no capítulo 4 depois de se realizar a parte experimental no capítulo 3.

No capítulo 3 começamos por desenvolver as bases necessárias à aplicação experimental que traduza o comportamento do sistema acoplado. O uso de circuitos para comprovar experimentalmente a teoria é algo que foi sempre aceite no meio científico, e serve de aplicação prática rápida. É apresentado um circuito e justificada a correspondência matemática a um oscilador de van der Pol, para o qual se definem as quantidades essenciais como amplitude, e são apresentados os resultados de simulação e experimentais para o circuito. Em seguida é construído um circuito de acoplamento que respeite a função matemática de acoplamento mostrada nos capítulos anteriores, chegando-se a um complicado modelo. Na última parte deste capítulo é apresentado o sistema completo, com dois osciladores e o acoplamento, onde se faz o paralelismo entre os resultados de simulação e a correspondente componente experimental. De notar que a base para tal construção foi sempre um modelo teórico de um oscilador autossustentado, e um modelo teórico de uma função de acoplamento, e que o objetivo era comprovar a base teórica apresentada nos capítulos 1 e 2, usando componentes físicas que não respeitam a 100% as expressões matemáticas, no entanto, o leitor poderá verificar que os resultados são muito satisfatórios.

No capítulo 4 confrontamos as ideias dos métodos teóricos e experimentais até então apresentados. O capítulo começa por um desenvolvimento das ferramentas essenciais para se verificar a correspondência da teoria com a prática. O essencial problema neste tópico foi a definição das variáveis e parâmetros da equação do oscilador de van der Pol como componentes do circuito acoplado. Depois de uma analise pormenorizada ao circuito, mostram-se as figuras de Lissajous que comprovam o fenómeno de sincronização e a ligação com a teoria.

No capítulo 5 é apresentado um estudo similar ao conjunto dos capítulos anteriores, mas desta feita para 3 osciladores. A razão de se mostrar este

caso particular baseia-se principalmente nos termos do acoplamento, que se modificam pelas alterações às partes teóricas e experimentais. São mostradas as equações para o método numérico e o de Krylov-Bogoliubov, tanto nos estados desacoplados como síncronos, e no fim do capítulo, é apresentado o procedimento experimental para 3 osciladores.

No capítulo 6 fechamos o nosso trabalho com a extensão do estudo a mais osciladores acoplados segundo primeiros vizinhos. Depois de uma introdução a grafos mais complexos, surge a integração de 10 unidades de forma numérica. Em seguida estuda-se o fenómeno de sincronização sujeito à variação dos parâmetros. É tomada especial atenção à distribuição de frequências naturais, ou seja, à heterogeneidade do sistema. Este capítulo serve também de conclusão, apesar de que depois o leitor pode encontrar um conjunto de anexos que servem de complementação ao apresentado durante a tese.

Palavras-Chave: Sincronização, van der Pol, Equações Diferenciais, Não Linear, Acoplamento, Circuitos

Chapter 1

Introduction

Nature performs complex phenomena in all scales. Scientists attempt to create theoretical, mathematical and computational models in order to know how these events work, and apply them to all subjects, from physics and biology to sociology and psychology. One of nature's complex phenomenon is the core of this very work - synchronization. Synchronization is a complex dynamical process [1], that describes the adjustment of rhythms, due to some coupling, between different objects that have some kind of oscillatory motion. This general definition encompasses a variety of systems in nature, like the synchronous periodical flash of an ensemble of fireflies (biological oscillators [2]), in man-made systems, like electrical and chemical engineering [3], or a combination of both, like neuroscience [4], robotics [5], among many others. In the last ten years, a great enthusiasm from the scientific community, and the general public, has emerged. Still, we have to go a few centuries back to 1660's to name its discoverer - Christiaan Huygens [6]. He noticed that two pendula clocks with a non-rigid common support connecting them, would synchronize.

The area of mathematics primarily involved in these studies is called dynamical systems. The oscillatory objects mentioned above are modelled as autonomous dynamical systems, specifically self-sustained oscillators. Their main characteristics are: an energy input to balance the dissipation in the oscillatory motion; no explicit time dependence - which implies that if $f(t)$ is a solution, $f(t+\xi)$ is also a solution, for all ξ; and a well-defined amplitude of the oscillations. If the system is perturbed, it returns to its stable trajectory after a small transient. This reason will allow us to avoid studying the behaviour and evolution of the amplitude of the dynamical system. Moreover, these properties mean that stable regime does not depend on initial conditions, and that the equations of motion have to be nonlinear - linear systems do not exhibit self-sustained oscillations. These self-sustained oscillations correspond to limit cycles in the phase space of the system, i.e., closed curves that represent the values taken by the variables associated with the degrees of freedom.

In this chapter we will discuss the fundamental ideas behind synchronization, and the coupling of oscillators. We start with a description of the van der Pol oscillator and its properties, and then move on to notions on coupling, topology of the dynamical system, and a method that ensures the reliability of the procedures of the following chapters - the $KBAM$.

1.1 Motivation

The main character of this thesis is the van der Pol oscillator that is modelled by the following equation:

$$\ddot{x}(t) - \mu\left(1 - x^2(t)\right)\dot{x}(t) + \omega^2 x(t) = 0, \tag{1.1}$$

which can also be written as:

$$\ddot{x}(t) - \left(\mu - x^2(t)\right)\dot{x}(t) + \omega^2 x(t) = 0, \tag{1.2}$$

with a change of variables $\widetilde{x}(t) = \frac{1}{\sqrt{\mu}}x(t)$ - their equivalence is shown in Appendix A. The parameters ω and μ are the natural frequency of the oscillator and a control parameter, respectively. The goal of this work is to show, with an intensive analytical, numerical and experimental study, that synchronization emerges when vdP oscillators are coupled. The choice of this particular oscillator is not important, since we draw attention to the nearly harmonic oscillations in the small non-linearity range.

1.1.1 The van der Pol Oscillator

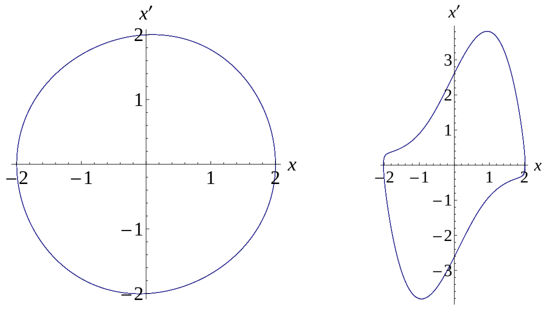
The van der Pol oscillator was introduced in the $1920's$ by Balthasar van der Pol $(1889 - 1959)$, and is an oscillator with nonlinear damping, expressed in equation 1.1, that exhibits self-sustained oscillations - a limit cycle in the phase space around its fixed point $(x, \dot{x}) = (0, 0)$ (see Appendix B).

C1 & ApC - Limit Cycle and Perturbative Solution.nb

According to Liénard's theorem, the vdP oscillator has a unique stable limit cycle, and a plot for $\mu = 0.05$ can be seen in figure 1.1a . Although in this work we emphasize quasi-harmonic oscillations, i.e., the regime for $\mu \ll 1$, the vdP oscillator is also known for its particular relaxation oscillations, i.e., for large μ - a corresponding plot of relaxation oscillations for $\mu = 2$ is shown in figure 1.1b.

1.1.2 Coupling

In many complex systems, the complexity arises from the interplay of large number of constituents parts, or units, whose isolated behaviour may be simple and well understood. The interaction between the different units of the system is represented, in strength and in form, by coupling terms. If the



(a) $\mu = 0.05$ **(b)** $\mu = 2$

Figure 1.1: Plot of a solution of the van der Pol vdP oscillator on the phase plane $(x, \dot{x})$.

coupling is too strong, the system will start to behave itself as a unit, and if the coupling is too weak, every unit of the system will behave independently from its neighbours. In both scenarios, synchronization is not possible, so it is very important to know the range of coupling intensity where each unit has its own dynamics, and is influenced by the dynamics of the ensemble. A good example that can be drawn from this definition is the synchronization of footsteps of two individuals, let us say A and B, walking side by side. We can assume the coupling is a complex function of the visual field of each, the sound of the footsteps, the distance between them, and probably many more variables. If we connect the left foot of A to the right foot of B with a rigid bar, they will perform motion in sync, but it will not be a synchronization phenomenon. On the other hand, if A and B walk at a considerable lateral distance, do not hear or see each other's footsteps, they will probably walk in their natural walking speed. Nevertheless, if the right properties are met, and if, for example, their height difference (meaning, the height of their legs, consequently, the amplitude of their footsteps) and hurry-to-get-to-destination difference is not overwhelming, then they will probably synchronize their walking speed and footsteps.

The coupling intensity is a value that quantifies the influence of one unit on another. Of course in the real-life case of footsteps, it is rather difficult to measure, meaning to express as a number, the intensity of the interaction. Still, here we will mention two coupling constants that represent the intensity of two different kinds of coupling. There are two different ways in which the units - the oscillators - influence each other, and we will show how we can define these terms.

1.2 Background

1.2.1 Topology

From the point of view of its inner structure, a system may be represented by a graph or network in which nodes are the units and links are the couplings. The simplest non-trivial example is the graph formed by two connected nodes - two oscillators represented by two nodes, with a simple coupling between them, represented by a link. This case will be considered in the next chapter.

In order to study the coupling function, we need to use three matrices: Adjacency, Degree and Kirchhoff matrices.

Adjacency Matrix

The adjacency matrix is the matrix whose elements a_{ij} are $1's$ or $0's$, depending on whether the node i has any link to node j, or not, respectively. If the links are just the connections between nodes, and do not imply any kind of preferable direction - unidirectional coupling - the adjacency matrix is symmetric. In the graph of figure 1.2, node 1 has a link to node 2, meaning that the coefficient $a_{12} = a_{21} = 1$. The feedback link on node 1 is represented, by standard convection, by $a_{11} = 1$. The double link (or in general the n-tuple link), between node 3 and node 5 is also represented by $a_{35} = a_{53} = 1$. Hence, the adjacency matrix A for this graph is:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \tag{1.3}$$

Degree Matrix

The degree matrix is a diagonal matrix that quantifies the number of links in each node. By convention, a feedback link is counted twice, so, for the graph in figure 1.2, the element a_{11} is $a_{11} = 3$: a feedback link (two links), plus a link to node 2 (one link). Considering this, the degree matrix D is:

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \tag{1.4}$$

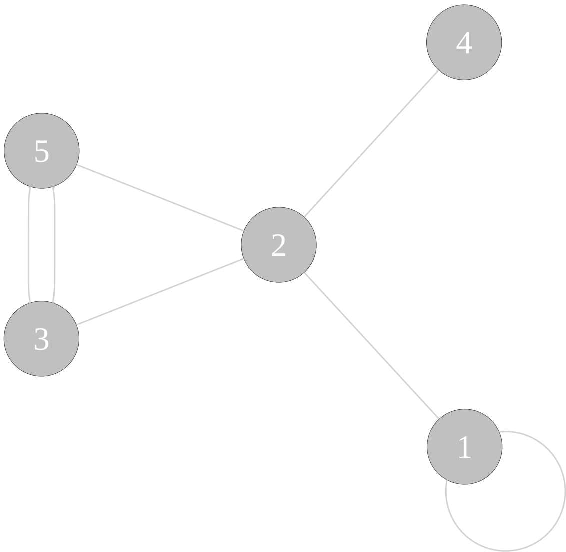


Figure 1.2: Example of a graph with 5 nodes.

Kirchhoff Matrix

The Kirchhoff matrix K, also called connectivity or admittance matrix, defines the nodes and connections of a graph. The matrix K is defined by the difference between the degree matrix D and the adjacency matrix A. In the case of figure 1.2 we write the Kirchhoff matrix K as:

$$K = D - A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 3 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 3 \end{bmatrix} \tag{1.5}$$

From this topological point of view, all nodes are similar - the networks do not represent the possibly different properties of the individual nodes of the system. We will choose vdP oscillators that are just slightly different.

1.2.2 Detuning

Detuning is a specific parameter that measures the heterogeneity of the system and represents the variation in the parameters of the different elements of the ensemble. It may be a little difficult to think of the detuning parameter in an ensemble of n oscillators, however, consider the natural frequency of each oscillator - we can define a frequency distribution that measures the range of frequencies, or a detuning vector composed of the detuning parameter associated with each pair of oscillators.

1.2.3 Phase

Phase is a variable that measures the fraction of the period corresponding to a certain periodic state. We are interested in the evolution of the phase difference for each pair, i.e., if it tends to a constant value the oscillators will be synchronized.

The amplitude of an autonomous oscillator is well-defined (see Appendix C), and hence its motion is completely described by the phase along the limit cycle. A different scenario is found for the phase. The phase can easily be changed by a longitudinal perturbation, which travels along the limit cycle. Denoting the phase by φ, and defining it in such a way that it changes uniformly along the limit cycle, we can write the equation for the phase as:

$$\frac{d\varphi}{dt} = \omega, \tag{1.6}$$

where $\omega = 2\pi/T$ is the frequency and T is the period.

The study of the evolution of the phase, and especially the phase difference between identical coupled units, has shown two significant stable points in a wide range of systems: Zero, denoted by in-phase synchrony, and π, denoted analogously by anti-phase synchrony. For instance, Christiaan Huygens noticed that the pendula he observed reached a state of anti-phase synchronization. Nevertheless, if we have a non-zero detuning parameter we may have slightly different values for the phase difference.

1.2.4 Coupling Function

The behaviour of a system of n uncoupled units is governed by a system of differential equations:

$$\dot{\vec{x}} = \vec{f}\left(\vec{x}\right), \tag{1.7}$$

in which $\vec{x}$ is defined as $\vec{x} = (x_1, \dot{x}_1, ..., x_n, \dot{x}_n)$, and $\vec{f}\left(\vec{x}\right)$ will have the form of the vdP equations. In addition, we will define an extra 2×2 matrix, the output matrix H, that will help us to construct and understand the coupling term. Basically, the output matrix H selects which terms of $\vec{x}$ we want to include in the coupling function. Since we want bidirectional coupling, both in amplitudes as in their derivatives, we will need two output functions H_1 and H_2, and also two coupling coefficients, so we can manipulate which coupling term is the most important by varying these coefficients.

The general form for adding a linear coupling in system 1.7 is due to Pecorra and Carrol [7], and states:

$$\dot{\vec{x}} = \vec{f}\left(\vec{x}\right) + \left(K \otimes (\sigma H)\right)\vec{x}, \tag{1.8}$$

in which σ is the coupling coefficient, $\otimes$ is the Kronecker direct product, and σH is defined as:

$$\sigma H = \sigma_1 H_1 + \sigma_2 H_2 + \ldots + \sigma_n H_n. \tag{1.9}$$

All other variables retain the previous definitions.

In our work we consider two types of coupling coefficients, one that is denoted by amplitude coupling C_A, and the other which is denoted by derivative coupling C_D. Therefore:

$$\sigma = (\sigma_1, \sigma_2) = (C_A, C_D), \tag{1.10}$$

is the coupling vector. Also, two output functions (H_1, H_2) to collect the right terms in the positions and derivatives. Thus, the general form of the system of n coupled units, with coupling function formed by the coupling terms C_A and C_D, and with output matrices H_1 and H_2, respectively, is:

$$\dot{\vec{x}} = \vec{f}(\vec{x}) + C_A (K \otimes H_1) \vec{x} + C_D (K \otimes H_2) \vec{x}. \tag{1.11}$$

Every output function may give us different coupling terms, so different results can emerge, which may, or may not, have physical meaning. The goal here is to build a theory based on mutual synchronization, and that will be our concern when defining these matrices. Unlike the case of external forced synchronization, where an autonomous oscillator synchronizes its rhythm with an external oscillating mechanism, mutual synchronization implies that the influence of all units have the same order of magnitude - bidirectional coupling - and is concerned with the study of oscillating objects that influence each other in this way.

1.3 Krylov-Bogoliubov Averaging Method

In order to study our mutual synchronization model we will use the Krylov-Bogoliubov averaging method [8], to derive an equation for the evolution of the phase difference. In this section, our goal is to prove that KB is appropriate for the type of systems under consideration in the following chapter. This method applies to systems in the form (please consider the simplified notation of the following equation as a system of any dimension you desire):

$$\ddot{x} + \omega^2 x + \mu f(x, \dot{x}) = 0, \tag{1.12}$$

where μ is a small parameter, so that $\mu f(x, \dot{x})$ is considered as a perturbation of the harmonic oscillator. Recalling the vdP equation, with $f(x, \dot{x}) = \left(x^2 - 1\right) \dot{x}$, we realise that the vdP oscillator can be represented by the family of equations 1.12.

We will start by considering the simplest case with $\mu = 0$. One solution for equation 1.12 is:

$$\begin{cases} x = A\cos(\omega t + \varphi) \\ \\ \dot{x} = -A\omega\sin(\omega t + \varphi) \end{cases} \tag{1.13}$$

where A and φ are constants that represent the amplitude and the phase, respectively. The corresponding solutions for A and φ time-dependent:

$$x = A(t)\cos(\omega t + \varphi(t)), \tag{1.14}$$

$$\dot{x} = \dot{A}(t)\cos(\omega t + \varphi(t)) - A(t)(\omega + \dot{\varphi}(t))\sin(\omega t + \varphi(t)), \tag{1.15}$$

are of more interest to us, because the method takes them as a basis to construct approximate solutions for the case with $\mu \neq 0$. In this work, the case with $\mu < 0$ is not considered, because it represents a chaotic regime, so we will just consider the case with $\mu > 0$ - self-sustained oscillations occur only in this regime.

1.3.1 First Approximation

In a n-order differential equation we have n degrees of freedom. In 1.12 we have two degrees of freedom which we explicit in the harmonic-like solutions of 1.13, and now we want to choose the terms from equation 1.15 that are different from $\dot{x}$ in equation 1.13, and equalize them to zero. Hence:

$$\dot{A}\cos(\omega t + \varphi) - A\dot{\varphi}\sin(\omega t + \varphi) = 0. \tag{1.16}$$

So:

$$\dot{x} = -A\omega\sin(\omega t + \varphi),$$

and the derivative $\ddot{x}$:

$$\ddot{x} = -\dot{A}\omega\sin(\omega t + \varphi) - A\omega(\omega + \dot{\varphi})\cos(\omega t + \varphi). \tag{1.17}$$

We substitute $\dot{x}$ and $\ddot{x}$ in the equation of motion 1.12:

$$-\dot{A}\omega\sin(\omega t + \varphi) - A\omega(\omega + \dot{\varphi})\cos(\omega t + \varphi) + \omega^2 A\cos(\omega t + \varphi) + \mu f(x, \dot{x}) = 0, \tag{1.18}$$

The terms with ω^2 cancel out. Adding condition 1.16 we can solve a system of two equations with two variables, A and φ:

$$\begin{aligned} &\dot{A}\cos(\omega t + \varphi) = A\dot{\varphi}\sin(\omega t + \varphi) \\ \\ &\dot{A}\omega\sin(\omega t + \varphi) + A\omega\dot{\varphi}\cos(\omega t + \varphi) = \mu f(x, \dot{x}) \end{aligned} \tag{1.19}$$

After some algebra:

$$
\begin{aligned}
\dot{A} &= \tfrac{\mu}{\omega} f\left(x, \dot{x}\right) \cos\left(\omega t + \varphi\right) \\
\dot{\varphi} &= \tfrac{\mu}{A\omega} f\left(x, \dot{x}\right) \sin\left(\omega t + \varphi\right)
\end{aligned}
\tag{1.20}
$$

They depend on the small parameter μ, hence are functions that vary slowly during one period $T = \frac{2\pi}{\omega}$. Writing them in a more convenient form using Fourier expansions for the second members (explicit Fourier coefficients in Appendix D - with $\tilde{\varphi} = \omega t + \varphi$):

$$
\begin{aligned}
f\left(x, \dot{x}\right) \cos\left(\tilde{\varphi}\right) &= K_0(A) + \textstyle\sum_{n=1}^{\infty} K_n(A) \cos(n\tilde{\varphi}) + L_n(A) \sin(n\tilde{\varphi}) \\
f\left(x, \dot{x}\right) \sin\left(\tilde{\varphi}\right) &= P_0(A) + \textstyle\sum_{n=1}^{\infty} P_n(A) \cos(n\tilde{\varphi}) + Q_n(A) \sin(n\tilde{\varphi})
\end{aligned}
,
\tag{1.21}
$$

equation 1.20 become:

$$
\begin{aligned}
\dot{A} &= \tfrac{\mu}{\omega} K_0\left(A\right) + \tfrac{\mu}{\omega} \textstyle\sum_{n=1}^{\infty} K_n\left(A\right) \cos\left(n\left(\omega t + \varphi\right)\right) + L_n\left(A\right) \sin\left(n\left(\omega t + \varphi\right)\right) \\
\dot{\varphi} &= \tfrac{\mu}{\omega A} P_0\left(A\right) + \tfrac{\mu}{\omega A} \textstyle\sum_{n=1}^{\infty} P_n\left(A\right) \cos\left(n\left(\omega t + \varphi\right)\right) + Q_n\left(A\right) \sin\left(n\left(\omega t + \varphi\right)\right)
\end{aligned}
\tag{1.22}
$$

For n integer:

$$
\int_{t}^{t+T} \cos\left(n\left(\omega t + \varphi\right)\right) dt = \int_{t}^{t+T} \sin\left(n\left(\omega t + \varphi\right)\right) dt = 0,
\tag{1.23}
$$

so, we can integrate $\dot{A} = \frac{dA}{dt}$ and $\dot{\varphi} = \frac{d\varphi}{dt}$:

$$
\int_{t}^{t+T} \dot{A} dt = \int_{t}^{t+T} \frac{\mu}{\omega} K_0\left(A\right) dt,
\tag{1.24}
$$

where the function being integrated on the right-hand-side is time-independent. Simple integration will lead us to:

$$
\frac{A(t+T) - A(t)}{T} = \frac{\mu}{\omega} K_0\left(A\right)
\tag{1.25}
$$

and doing the same to $\dot{\varphi}$:

$$
\frac{\varphi(t+T) - \varphi(t)}{T} = \frac{\mu}{\omega A} K_0(A)
\tag{1.26}
$$

Now, we recall that ΔA and $\Delta\varphi$ do not vary much in the interval $[t, t+T]$ due to μ being small, we can write these expressions as derivatives:

$$
\begin{aligned}
\tfrac{dA}{dt} &= \tfrac{\mu}{\omega} K_0(A) \\
\tfrac{d\varphi}{dt} &= \tfrac{\mu}{\omega A} P_0(A)
\end{aligned}
\tag{1.27}
$$

Comparing these equations with the 1.22, we conclude that equations 1.27 are the result, by 1.23, of averaging the former over one period.

In the next chapter we will introduce the van der Pol coupled system and solve it analytically and numerically. Understanding the basic ideas behind the Krylov-Bogoliubov method is essential, in a way that the time average is the crucial step to obtain the analytical approximated solutions. It is trivial to notice that for systems of the type of equation 1.12, for small μ, both the amplitude and the phase will vary slowly (μ - *dependent*) in one period, allowing us to perform simplifications without losing much information. We will combine this approximate result with a polar coordinate transformation, aiming at deriving the equations for the evolution of the amplitude and the phase. Our purpose is to prove that the approximated method is appropriate for our system, and that systems of this type can be coupled to perform synchronized motion.

After completing the theoretical and experimental studies for two coupled oscillators, we will use these general considerations to perform an extension to this work by increasing the number of oscillators, in Chapters 5 and 6.

Chapter 2

Synchronization

After the discovery by Huygens, more and more thinkers became interested in the ideas of organization, or lack of it. One of them, rising in the early twentieth century, was *William Gibbs*, who described the phenomenon of entropy as *Mixed-up-ness*. The common idea of entropy tell us that its variation in an isolated system will never decrease. Adding the idea of disorder to the entropy one (as is usually attributed), it makes a major milestone in science, leading to huge discussions whenever someone tries to talk about the natural organization or decrease of entropy. However, the universe is not a completely random set of particles wandering the vastness of the darkness. There are galaxies, solar systems, and living beings, like us humans. Somehow order wins the battle of nature's complexity, and natural organization emerges everywhere. Wherever we look, we see the pulsating rhythm of nature, at all scales, with its melody and tempo, like synchronization is an intrinsic maestro of the universe.

Synchronization is a natural process that any person can understand. A set of different individuals in interaction will perform their work with lower losses of energy if they perform it in unison, i.e., two people walking side by side in the street will be very tired (at least one of them) if they walk in their natural speed, and then, every few seconds, stop or speed up to catch the other one, rather than if both walk in an average speed. The idea is simple and the conclusions are very interesting, mainly when we can observe some event where, unexpectedly, different units start to behave as a group, like the previously mentioned flash of fireflies.

In this chapter our goal is to provide the basic mathematical knowledge of how to look for mutual synchronization of two coupled oscillators. In the next chapters we will need these notions to perform the study thereafter, so it is important, for the reader, to know the basics herein.

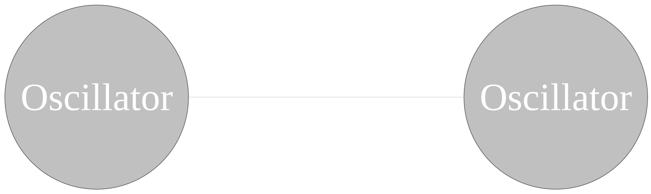


Figure 2.1: Graph with two nodes (vertices) and a line (edge). This graph represents a one-on-one coupling between two oscillators.

2.1 Equation of Motion for Two Oscillators

In this section we will define the quantities introduced in Chapter 1, considering the simplest case of coupled oscillators. A review on the topology, output functions, and coupling terms will be presented, in order to derive an equation for the motion of the system.

2.1.1 Topology

C2 - Coupling Term 2vdp.nb

Two coupled oscillators are represented by the simple graph with two nodes with one link in figure 2.1. The matrices will be denoted with the index *2vdp* for this particular case. The degree matrix is:

$$D_{2vdp} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.1}$$

meaning each node have only one connection. The adjacency matrix is simply:

$$A_{2vdp} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.2}$$

since there is no feedback in each node, and there is only one connection between them. Finally, the Kirchhoff matrix is:

$$K_{2vdp} = D_{2vdp} - A_{2vdp} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{2.3}$$

2.1.2 Output Functions

In Chapter 1 we said that the output functions represent which variables (x,$\dot{x}$,$\ddot{x}$,...) participate in the coupling terms. We will define output functions that are logical for two oscillators, and use the same for all the other cases. For two oscillators, it is straightforward to use coupling terms that depend on the difference between the positions of each, and on the difference of the derivatives of the positions of each. So, recalling equation 1.11, we may write:

$$K_{2vdp} \otimes (\sigma H)\,, \tag{2.4}$$

with $\sigma = (C_A, C_D)$ and $H = (H_1, H_2)$. Hence (calculating only the coupling for the amplitude difference):

$$C_A \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \begin{bmatrix} h1_{11} & h1_{12} \\ h1_{21} & h1_{22} \end{bmatrix}, \tag{2.5}$$

which results in:

$$C_A \begin{bmatrix} h1_{11} & h1_{12} & -h1_{11} & -h1_{12} \\ h1_{21} & h1_{22} & -h1_{21} & -h1_{22} \\ -h1_{11} & -h1_{12} & h1_{11} & h1_{12} \\ -h1_{21} & -h1_{22} & h1_{21} & h1_{22} \end{bmatrix}. \tag{2.6}$$

As we said, we want coupling terms that represent the difference between amplitudes (position),

$$C_A \begin{bmatrix} h1_{11} & h1_{12} & -h1_{11} & -h1_{12} \\ h1_{21} & h1_{22} & -h1_{21} & -h1_{22} \\ -h1_{11} & -h1_{12} & h1_{11} & h1_{12} \\ -h1_{21} & -h1_{22} & h1_{21} & h1_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ C_A\,(x_2 - x_1) \\ 0 \\ C_A\,(x_1 - x_2) \end{bmatrix}. \tag{2.7}$$

Solving this system, we will end up with $h1_{11} = 0$, $h1_{12} = 0$, $h1_{21} = -1$ and $h1_{22} = 0$:

$$H_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}. \tag{2.8}$$

Analogously to the case of the difference in their derivatives, we get:

$$H_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.9}$$

2.1.3 Mutual Synchronization

Substituting all quantities in equation 1.11, it results in the system of coupled oscillators:

$$\begin{aligned} \dot{x}_1 &= y_1 \\ \dot{y}_1 &= -\mu_1(x_1^2 - 1)\dot{x}_1 - \omega_1^2 x_1 - C_A\,(x_1 - x_2) - C_D\,(\dot{x}_1 - \dot{x}_2) \\ \dot{x}_2 &= y_2 \\ \dot{y}_2 &= -\mu_2(x_2^2 - 1)\dot{x}_2 - \omega_2^2 x_2 - C_A\,(x_2 - x_1) - C_D\,(\dot{x}_2 - \dot{x}_1) \end{aligned}\,, \tag{2.10}$$

or in a more compact form:

$$\begin{aligned} &\ddot{x}_1 + \mu_1(x_1^2 - 1)\dot{x}_1 + \omega_1^2 x_1 + C_A\,(x_1 - x_2) + C_D\,(\dot{x}_1 - \dot{x}_2) = 0 \\ &\ddot{x}_2 + \mu_2(x_2^2 - 1)\dot{x}_2 + \omega_2^2 x_2 + C_A\,(x_2 - x_1) + C_D\,(\dot{x}_2 - \dot{x}_1) = 0 \end{aligned} \quad . \tag{2.11}$$

The coupling terms depend on the difference of x_1, x_2 and $\dot{x}_1, \dot{x}_2$, so whenever $x_1 = x_2$ and/or $\dot{x}_1 = \dot{x}_2$, the respective coupling is zero. Moreover, the larger the difference between them, the larger the influence of the coupling, and we take this monotonous dependence to be linear.

In a system of only two units, detuning may be the difference between the natural frequencies of each. This information tells us how different the oscillators are at time $t = 0$, and obviously, if the detuning is small, then synchronization will occur easier than if detuning is large.

2.2 Phase Difference

We will derive an equation for the evolution of the phase difference of two coupled vdP oscillators. The usage of the $KBAM$ is justified by two main points: the fact that almost every time it is impossible to obtain analytical results without approximations; and the fact that in real life, we are only concerned with a certain range of the parameters. Certainly, solutions for the whole domain are more general and if possible, are preferable, but they are also unnecessary if you are only concerned with an narrow range of possible values for the considered parameters.

In Chapter 1 we have seen that the nonlinear vdP oscillator 1.1 undertake quasi-harmonic oscillations with μ a small positive real number. Hence, we will start by assuming we solve 2.11 with a perturbation of the harmonic solution:

$$x_i(t) = A_i(t)\cos\left(\omega_s t + \varphi_i(t)\right), \qquad i = 1, 2 \tag{2.12}$$

where A is the amplitude and φ is the phase, which are allowed to vary in time, and ω_s is the synchronization frequency, i.e., the frequency after the synchrony transient, so we are assuming that eventually the system will synchronize, and the final common frequency is ω_s.

2.2.1 Quasi-Harmonic Solutions

The two oscillators are experiencing two different couplings: one in their amplitudes - C_A - and one of the derivative of the amplitudes - C_D [9]. The equations of motion are (from now on, we are omitting the temporal dependence of the functions):

$$\begin{cases} \ddot{x}_1 - \mu_1(1 - x_1^2)\dot{x}_1 + \omega_1^2 x_1 + C_A\,(x_1 - x_2) + C_D\,(\dot{x}_1 - \dot{x}_2) = 0 \\ \ddot{x}_2 - \mu_2(1 - x_2^2)\dot{x}_2 + \omega_2^2 x_2 + C_A\,(x_2 - x_1) + C_D\,(\dot{x}_2 - \dot{x}_1) = 0 \end{cases} \tag{2.13}$$

where the parameters are: $\mu_{1,2}$ - non-linearity control parameters, $\omega_{1,2}$ - natural frequencies. Nonetheless, until we need to use the system in this form, we will proceed the derivation just for one oscillator, because the derivation is analogous for the other. The indices through the derivation will be omitted on any indubitable step. We will be working with:

$$\ddot{x}_1 - \mu_1(1 - x_1^2)\dot{x}_1 + \omega_1^2 x_1 + C_A\,(x_1 - x_2) + C_D\,(\dot{x}_1 - \dot{x}_2) = 0. \tag{2.14}$$

Starting with the harmonic solution, we use Euler's Identity to write equation 2.12 as an exponential form:

$$x = A\cos(\omega_s t + \varphi) = \frac{1}{2}\left(a\,e^{i\omega_s t} + a^* e^{-i\omega_s t}\right), \tag{2.15}$$

where, basically, we use a change of variables:

$$\begin{cases} a = A\,e^{i\varphi} \\ a^* = A\,e^{-i\varphi} \end{cases} \tag{2.16}$$

to change from, so to speak, a trigonometric form to an exponential form.

Writing the first derivative of the trigonometric form of equation 2.15:

$$\dot{x} = \dot{A}\cos(\omega_s t + \varphi) - A\,(\omega_s + \dot{\varphi})\sin(\omega_s t + \varphi)\,, \tag{2.17}$$

and in order to relate to the KB procedure in section 1.3, we will choose:

$$\dot{A}\cos(\omega_s t + \varphi) - A\dot{\varphi}\sin(\omega_s t + \varphi) = 0, \tag{2.18}$$

which results in:

$$\dot{x} = -A\,\omega_s\sin(\omega_s t + \varphi) = \frac{i\omega_s}{2}\left(a\,e^{i\omega_s t} - a^* e^{-i\omega_s t}\right). \tag{2.19}$$

The decision of making equation 2.18 null comes from the degree of freedom we have from the conditions of the differential equation.

Differentiating 2.19:

$$\begin{aligned} \ddot{x} &= \tfrac{i\omega_s}{2}\left(\dot{a}\,e^{i\omega_s t} + a\,i\omega_s e^{i\omega_s t} - \dot{a}^* e^{-i\omega_s t} + i\omega_s a^* e^{-i\omega_s t}\right) \\ &= \tfrac{i\omega_s}{2}\underbrace{\left(\dot{a}\,e^{i\omega_s t} - \dot{a}^* e^{-i\omega_s t}\right)}_{1)} - \tfrac{\omega_s^2}{2}\left(a\,e^{i\omega_s t} + a^* e^{-i\omega_s t}\right) \end{aligned} \tag{2.20}$$

Simplifying 1):

$$\begin{aligned} \dot{a}\,e^{i\omega_s t} - \dot{a}^* e^{-i\omega_s t} &= \dot{a}\,e^{i\omega_s t} - \dot{a}^* e^{-i\omega_s t} + \dot{a}\,e^{i\omega_s t} - \dot{a}\,e^{i\omega_s t} \\ &= 2\dot{a}\,e^{i\omega_s t} - \underbrace{\left(\dot{a}^* e^{-i\omega_s t} + \dot{a}\,e^{i\omega_s t}\right)}_{2)} \end{aligned} \tag{2.21}$$

and because of equation 2.18, the terms in 2) are zero (we show this relation in detail in Appendix E), resulting in the expression for 1):

$$\dot{a}\, e^{i\omega_s t} - \dot{a}^* e^{-i\omega_s t} = 2\dot{a}\, e^{i\omega_s t}, \tag{2.22}$$

So, equation 2.20 becomes:

$$\ddot{x} = i\omega_s \dot{a}\, e^{i\omega_s t} - \frac{\omega_s^2}{2}\left(a\, e^{i\omega_s t} + a^* e^{-i\omega_s t}\right). \tag{2.23}$$

2.2.2 Derivation of the Phase Difference Equation

Now we just substitute 2.15, 2.19 and 2.23 in 2.14. Recalling the previous steps, we have the equation of motion (for the oscillator 1):

$$\ddot{x}_1 - \mu_1(1 - x_1^2)\dot{x}_1 + \omega_1^2 x_1 + C_A\,(x_1 - x_2) + C_D\,(\dot{x}_1 - \dot{x}_2) = 0, \tag{2.24}$$

and the equations:

$$\begin{aligned} x =&\quad A\cos(\omega_s t + \varphi) = \tfrac{1}{2}\left(a\, e^{i\omega_s t} + a^* e^{-i\omega_s t}\right) \\ \dot{x} =&\quad \tfrac{i\omega_s}{2}\left(a\, e^{i\omega_s t} - a^* e^{-i\omega_s t}\right) \\ \ddot{x} =&\quad i\omega_s \dot{a}\, e^{i\omega_s t} - \tfrac{\omega_s^2}{2}\left(a\, e^{i\omega_s t} + a^* e^{-i\omega_s t}\right). \end{aligned} \tag{2.25}$$

After substitution, we will get:

$$\begin{aligned} & i\omega_s \dot{a}_1\, e^{i\omega_s t} - \tfrac{\omega_s^2}{2}\underbrace{\left(a_1\, e^{i\omega_s t} + a_1^* e^{-i\omega_s t}\right)}_{3)} - \\ & -\mu_1\left(1 - \left(\tfrac{1}{2}\left(a_1\, e^{i\omega_s t} + a_1^* e^{-i\omega_s t}\right)\right)^2\right)\tfrac{i\omega_s}{2}\left(a_1\, e^{i\omega_s t} - a_1^* e^{-i\omega_s t}\right) + \\ & +\tfrac{1}{2}\omega_1^2\underbrace{\left(a_1\, e^{i\omega_s t} + a_1^* e^{-i\omega_s t}\right)}_{4)} + \\ & +C_A\left(\tfrac{1}{2}\left(a_1\, e^{i\omega_s t} + a_1^* e^{-i\omega_s t}\right) - \tfrac{1}{2}\left(a_2\, e^{i\omega_s t} + a_2^* e^{-i\omega_s t}\right)\right) + \\ & +C_D\left(\tfrac{i\omega_s}{2}\left(a_1\, e^{i\omega_s t} - a_1^* e^{-i\omega_s t}\right) - \tfrac{i\omega_s}{2}\left(a_2\, e^{i\omega_s t} - a_2^* e^{-i\omega_s t}\right)\right) = 0. \end{aligned} \tag{2.26}$$

Collecting the terms in ω_s^2 and ω_1^2 which are 3) and 4), and factorizing the parenthesis in μ_1, will lead us to:

$$\begin{aligned}
&i\omega_s \dot{a}_1\, e^{i\omega_s t} + \tfrac{\omega_1^2-\omega_s^2}{2}\left(a_1\, e^{i\omega_s t} + a_1^* e^{-i\omega_s t}\right) - \\
&-\mu_1 \tfrac{i\omega_s}{2}\left(a_1\, e^{i\omega_s t} - a_1^* e^{i\omega_s t}\right) + \\
&+\tfrac{\mu_1 i\omega_s}{8}(a_1^3 e^{3i\omega_s t} - a_1^2 a_1^* e^{i\omega_s t} + \left(a_1^*\right)^2 a_1 e^{-i\omega_s t} - \\
&-\left(a_1^*\right)^3 e^{-3i\omega_s t} + 2a_1^2 a_1^* e^{i\omega_s t} - 2a_1\left(a_1^*\right)^2 e^{-i\omega_s t}) \\
&+\tfrac{C_A}{2}\left(a_1\, e^{i\omega_s t} + a_1^* e^{-i\omega_s t} - a_2\, e^{i\omega_s t} - a_2^* e^{-i\omega_s t}\right) + \\
&+\tfrac{i\omega_s C_D}{2}\left(a_1\, e^{i\omega_s t} - a_1^* e^{-i\omega_s t} - a_2\, e^{i\omega_s t} + a_2^* e^{-i\omega_s t}\right) = 0.
\end{aligned} \tag{2.27}$$

Multiplying the resulting equation by $\frac{e^{-i\omega_s t}}{i\omega}$:

$$\begin{aligned}
&\dot{a}_1 + \tfrac{\omega_1^2-\omega_s^2}{2i\omega_s}\left(a_1 + a^* e^{-2i\omega_s t}\right) - \tfrac{\mu_1}{2}\left(a_1 - a_1^* e^{-2i\omega_s t}\right) + \\
&+\tfrac{\mu_1}{8}(a_1^3\, e^{2i\omega_s t} - a_1^2 a_1^* + \left(a_1^*\right)^2 a_1\, e^{-2i\omega_s t} - \\
&-\left(a_1^*\right)^3 e^{-4i\omega_s t} + 2a_1^2 a_1^* - 2a_1\left(a_1^*\right)^2 e^{-2i\omega_s t}) + \\
&+\tfrac{C_A}{2i\omega_s}\left(a_1 + a_1^* e^{-2i\omega_s t} - a_2 - a_2^* e^{-2i\omega_s t}\right) + \\
&+\tfrac{C_D}{2}\left(a_1 - a_1^* e^{-2i\omega_s t} - a_2 + a_2^* e^{-2i\omega_s t}\right) = 0.
\end{aligned} \tag{2.28}$$

As we showed in the KB method in section 1.3, here we are allowed to kill all $e^{ni\omega_s t}$ terms, so after averaging:

$$\dot{a}_1 + \frac{\omega_1^2 - \omega_s^2}{2i\omega_s} a_1 - \frac{\mu_1}{2} a_1 + \frac{\mu_1}{8} a_1^2 a_1^* + \frac{C_A}{2i\omega_s}\left(a_1 - a_2\right) + \frac{C_D}{2}\left(a_1 - a_2\right) = 0. \tag{2.29}$$

Using the inverse relation of amplitudes accordingly to the previous transformation:

$$\begin{aligned}
&a = A\, e^{i\varphi} \\
&aa^* = |a|^2 \\
&a^2 a^* = A^3 e^{i\varphi}
\end{aligned} \tag{2.30}$$

Results in:

$$\begin{aligned}
&\dot{A}_1 e^{i\varphi_1} + i\dot{\varphi}_1 A_1 e^{i\varphi_1} + \tfrac{\omega_1^2-\omega_s^2}{2i\omega_s} A_1 e^{i\varphi_1} - \tfrac{\mu_1}{2} A_1 e^{i\varphi_1} + \tfrac{\mu_1}{8} A_1^3 e^{i\varphi_1} + \\
&+\tfrac{C_A}{2i\omega_s}\left(A_1 e^{i\varphi_1} - A_2 e^{i\varphi_2}\right) + \tfrac{C_D}{2}\left(A_1 e^{i\varphi_1} - A_2 e^{i\varphi_2}\right) = 0
\end{aligned} \tag{2.31}$$

Multiplying by $e^{-i\varphi_1}$, and separating real $\mathcal{R}$ and imaginary $\mathcal{I}$ parts, using the Euler identity, while we define the phase difference $\delta\varphi = \varphi_2 - \varphi_1$:

$$\begin{aligned}
\mathcal{R}_1 \quad &\rightarrow \quad \dot{A}_1 - \tfrac{\mu_1}{2}A_1 + \tfrac{\mu_1}{8}A_1^3 - \tfrac{C_A}{2\omega_s}A_2\sin(\delta\varphi) + \tfrac{C_D}{2}\left(A_1 - A_2\cos(\delta\varphi)\right) = 0 \\
\mathcal{I}_1 \quad &\rightarrow \quad \dot{\varphi}_1 A_1 + \tfrac{\omega_s^2-\omega_1^2}{2\omega_s}A_1 - \tfrac{C_A}{2\omega_s}\left(A_1 - A_2\cos(\delta\varphi)\right) - \tfrac{C_D}{2}\left(A_2\sin(\delta\varphi)\right) = 0
\end{aligned} \tag{2.32}$$

and analogously, we obtain for the second oscillator:

$$\begin{aligned}
\mathcal{R}_2 \quad &\rightarrow \quad \dot{A}_2 - \tfrac{\mu_2}{2}A_2 + \tfrac{\mu_2}{8}A_2^3 + \tfrac{C_A}{2\omega_s}A_1\sin(\delta\varphi) + \tfrac{C_D}{2}\left(A_2 - A_1\cos(\delta\varphi)\right) = 0 \\
\mathcal{I}_2 \quad &\rightarrow \quad \dot{\varphi}_2 A_2 + \tfrac{\omega_s^2-\omega_2^2}{2\omega_s}A_2 - \tfrac{C_A}{2\omega_s}\left(A_2 - A_1\cos(\delta\varphi)\right) + \tfrac{C_D}{2}\left(A_1\sin(\delta\varphi)\right) = 0
\end{aligned} \tag{2.33}$$

Knowing that synchronization phenomena do not occur for large frequency detuning parameters, we can assume that:

$$\omega_1 \sim \omega_2 \sim \omega_s \tag{2.34}$$

hence, we define:

$$\omega_1 + \omega_2 := 2\omega_s \tag{2.35}$$

and we can define a new variable:

$$\Delta := \frac{\omega_2^2 - \omega_1^2}{2\omega_s} = \omega_2 - \omega_1. \tag{2.36}$$

Finally, we divide the imaginary parts of 2.32 and 2.33 by A_1 and A_2 respectively, and write the equation of the evolution of the phase difference $\dot{\varphi}_2 - \dot{\varphi}_1 = \dot{\delta\varphi}$:

$$\dot{\delta\varphi} = \Delta + \frac{C_A}{2\omega_s}\cos(\delta\varphi)\left(\frac{A_2}{A_1} - \frac{A_1}{A_2}\right) - \frac{C_D}{2}\sin(\delta\varphi)\left(\frac{A_2}{A_1} + \frac{A_1}{A_2}\right), \tag{2.37}$$

and the equations for the amplitudes:

$$\begin{aligned}
\dot{A}_1 &= \tfrac{\mu_1}{2}A_1 - \tfrac{\mu_1}{8}A_1^3 + \tfrac{C_A}{2\omega_s}A_2\sin(\delta\varphi) - \tfrac{C_D}{2}\left(A_1 - A_2\cos(\delta\varphi)\right) \\
\dot{A}_2 &= \tfrac{\mu_2}{2}A_2 - \tfrac{\mu_2}{8}A_2^3 - \tfrac{C_A}{2\omega_s}A_1\sin(\delta\varphi) - \tfrac{C_D}{2}\left(A_2 - A_1\cos(\delta\varphi)\right)
\end{aligned} \tag{2.38}$$

2.3 Numerical Study

For the numerical study, we wanted to analyse the evolution of the phase difference in time, directly from the system:

$$\begin{aligned}
\ddot{x}_1 - \mu_1(1 - x_1^2)\dot{x}_1 + \omega_1^2 x_1 + C_A(x_1 - x_2) + C_D(\dot{x}_1 - \dot{x}_2) = 0 \\
\ddot{x}_2 - \mu_2(1 - x_2^2)\dot{x}_2 + \omega_2^2 x_2 + C_A(x_2 - x_1) + C_D(\dot{x}_2 - \dot{x}_1) = 0
\end{aligned} \tag{2.39}$$

and to do that, we had to write the system in the form:

$$\begin{aligned}
&\dot{x}_1 = y_1 \\
&\dot{y}_1 = \mu_1(1 - x_1^2)y_1 - \omega_1^2 x_1 - C_A\left(x_1 - x_2\right) - C_D\left(y_1 - y_2\right) \\
&\dot{x}_2 = y_2 \\
&\dot{y}_2 = \mu_2(1 - x_2^2)y_2 - \omega_2^2 x_2 - C_A\left(x_2 - x_1\right) - C_D\left(y_2 - y_1\right) = 0
\end{aligned} \tag{2.40}$$

and operate the substitutions $(i = 1, 2)$:

$$\begin{aligned}
x_i\left(t\right) &= \mathcal{A}_i \cos\left(\phi_i\right) \\
y_i\left(t\right) &= \mathcal{A}_i \sin\left(\phi_i\right)
\end{aligned} \tag{2.41}$$

and:

$$\begin{aligned}
\dot{x}_i\left(t\right) &= \dot{\mathcal{A}}_i \cos\left(\phi_{i1}\right) - \mathcal{A}\dot{\phi}_i \sin\left(\phi_i\right) \\
\dot{y}_i\left(t\right) &= \dot{\mathcal{A}}_i \sin\left(\phi_i\right) + \mathcal{A}\dot{\phi}_i \cos\left(\phi_i\right)
\end{aligned} \tag{2.42}$$

With equations 2.41 and 2.42 we will be able to find equations for the evolution of the amplitude and the phase, again after some simple calculations:

$$\begin{aligned}
\dot{\mathcal{A}}_i &= \cos\left(\phi_i\right)\dot{x}_i + \sin\left(\phi_1\right)\dot{y}_i \\
\dot{\phi}_i &= \tfrac{1}{\mathcal{A}_i}\left(\cos\left(\phi_i\right)\dot{y}_i - \sin\left(\phi_i\right)\dot{x}_i\right)
\end{aligned} \tag{2.43}$$

Substituting 2.40 in 2.43 and doing the necessary simplifications, we will get symmetric equations for $\dot{\mathcal{A}}_{1,2}$ and $\dot{\phi}_{1,2}$. The results are shown in the following equations (where the notation chosen is: when $i = 1$, $j = 2$ and vice-versa):

$$\begin{aligned}
\dot{\mathcal{A}}_i = \quad & \mathcal{A}_i \cos\left(\phi_i\right)\sin\left(\phi_i\right)\left(1 - \omega_i^2\right) + \mu_i\left(1 - \mathcal{A}_i^2\cos^2\left(\phi_i\right)\right)\mathcal{A}_i \sin^2\left(\phi_i\right) - \\
& -C_A \sin\left(\phi_i\right)\left(\mathcal{A}_i\cos\left(\phi_i\right) - \mathcal{A}_j\cos\left(\phi_j\right)\right) - \\
& -C_D \sin\left(\phi_i\right)\left(\mathcal{A}_i\sin\left(\phi_i\right) - \mathcal{A}_j\sin\left(\phi_j\right)\right)
\end{aligned} \tag{2.44}$$

$$\begin{aligned}
\dot{\phi}_i = \quad & -\sin^2\left(\phi_i\right) + \mu_i\left(1 - \mathcal{A}_i^2\cos^2\left(\phi_i\right)\right)\sin\left(\phi_i\right)\cos\left(\phi_i\right) - \omega_i^2\cos^2\left(\phi_i\right) - \\
& -C_A \cos\left(\phi_i\right)\left(\cos\left(\phi_i\right) - \tfrac{\mathcal{A}_j}{\mathcal{A}_i}\cos\left(\phi_j\right)\right) - \\
& -C_D \cos\left(\phi_i\right)\left(\sin\left(\phi_i\right) - \tfrac{\mathcal{A}_j}{\mathcal{A}_i}\sin\left(\phi_j\right)\right)
\end{aligned} \tag{2.45}$$

2.4 Resume of the Chapter

In this chapter our goal was to present the mathematical background for two coupled oscillators, how to obtain an approximated equation for the phase difference using the KB method, and analytical equations for the amplitudes and phases of the oscillators. The mathematical background here depicted will support future work, since the study of two interacting units is the fundamental dynamical approach to the dynamics of a system - pairwise interaction.

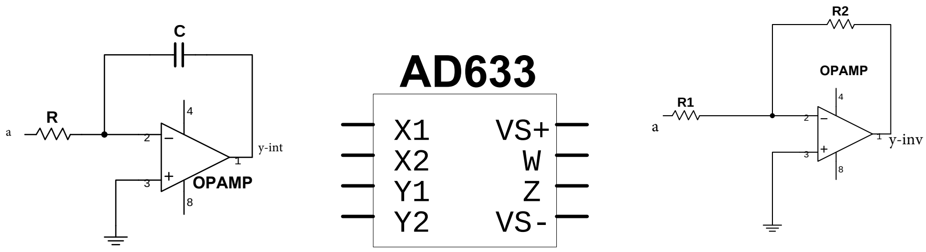
In Chapter 3, an analogous study from the experimental point of view, is presented, using tools from electronics to analyse the coupling of oscillators. Later, in Chapter 4, we will show the results for Chapters 2 and 3.

Chapter 3

Coupling Circuits

Times when scientists used pendula for almost everything are over. Now they look into electrical circuits as the future of mathematical engineering, in a way of creating physical structures that could behave like the mathematical equations. Sure it is impossible to physically reproduce the mathematical equations with zero error, but the effort of our ancestors led us to believe that working towards the reduction of the error is worthy. Hence, from the past one hundred years, electronics have evolved from an idea, to something indispensable in our everyday life - televisions, phones, cars, medical instruments, security control systems, and a never ending list of objects one avoids the need to think about its work and invention, although some of them are used all the time.

Coupling circuits, just as mentioned before, is also a powerful tool in electronics. A group of circuits working together to perform a task is incorporated basically almost in everything. One of the simplest examples is the transformer, which is used to charge, or power, the objects. The transformer is plugged into the socket and electrical current enters, travelling in an inductor which creates a magnetic field. The inductor is made of a certain number of coils that influence the range of the magnetic field. This allows to place another inductor in the range of the first one's magnetic field, which induces a current in the second inductor. This is called magnetic coupling. The current that enters through the plug do not reach your device, instead, both inductors are in magnetic coupling and the current created in the second one is the one that charges your device. Obviously, this is not a case of synchronization. When you unplug the transformer, the magnetic field vanishes after a transient, and therefore so does the current. The inductors are not self-sustained components. This is a perfect example of a false synchronization mechanism. Nevertheless, it helps to understand what sync is, and what is not, and it also points to an idea of coupling circuits. In this chapter we will develop a form of self-sustained electronic oscillators, with an electronic coupling.



(a) Inverting Integrator **(b)** Multiplier **(c)** Inverting Amplifier

Figure 3.1: Parts of the vdP Circuit

3.1 The van der Pol Oscillator

The first idea when doing an experiment of synchronized dynamics is to build an oscillating object that can reproduce, as reliable as possible, the differential equation that describes the motion, i.e., a physical object whose behaviour can be approximated by an autonomous dynamical system. We will describe a model of the vdP oscillator that is reported by Ned Corron in [10], and is shown in figure 3.2. This circuit has a very simple idea behind it - we need a circuit that represents a second order differential equation, and we need a nonlinear term which is quadratic. So, this circuit has implemented Inverting Integrators (figure 3.1a) and Multipliers (figure 3.1b). Starting at the potential point x, we will find that we can write the terms in $\ddot{x}$, $\dot{x}$, x and x^2. The components of the circuit will represent the control parameter and the frequency. Another part needed is an Inverting Amplifier (figure 3.1c) to change the sign of the potential at that point, and control the amplitude of the oscillations. The $AD633$ Multipliers $U1$ and $U2$, specified in figure 3.1b, have an output function W of:

$$W = \frac{(x1 - x2)\,(y1 - y2)}{10}, \tag{3.1}$$

whilst Z connects to a zero potential (ground), and $VS+$ and $VS-$ are the voltage inputs (± 15 respectively). The output function of the Inverting Integrator is (see Appendix F for the output functions of all parts used in this thesis):

$$y_{int} = -\frac{1}{C}\int \frac{a}{R} dt, \tag{3.2}$$

and the one for the Inverting Amplifier is:

$$y_{inv} = -\frac{R_2}{R_1} a. \tag{3.3}$$

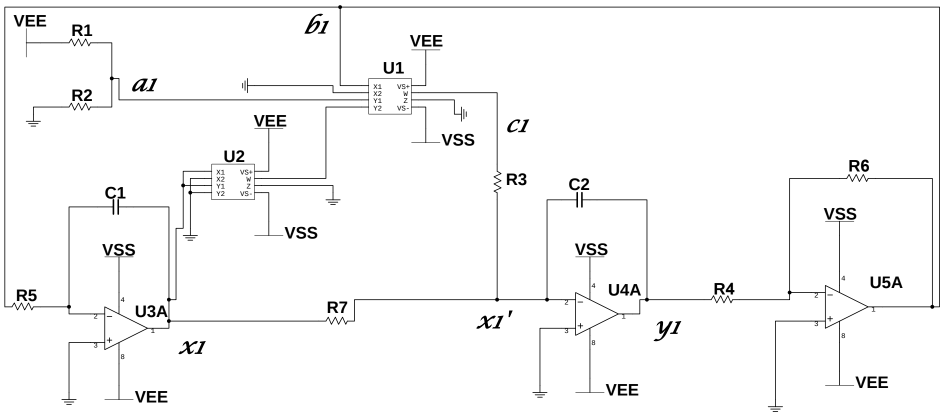


Figure 3.2: Circuit model of the vdP oscillator, representing the oscillator 1.

Note: The reference numbers of the electrical components in this chapter's figures, and in Appendix F do not correspond to the reference numbers in the equations of motion, which will follow directly from the figures referent to oscillator 1 and 2, and the correspondent figure for the coupling structure. In other figures, and in the Appendix, the goal is to show how to obtain the output function of all different parts. For example, the Inverting Differentiator in figure 3.4b has as components a capacitor C and a resistor R. In the figure of oscillator 1 there are two Inverting Differentiators, and the left one on the figure has as components C_1 and R_5.

3.1.1 Voltage Equation

Consider the potential points $\{a_1, b_1, c_1, x_1, y_1, x_1'\}$ in figure 3.2. At point x_1, we have:

$$x_1 = -\frac{1}{R_5 C_1}\int b_1 dt. \tag{3.4}$$

From the Inverting Amplifier at y_1 we have:

$$y_1 = -b_1 \frac{R_4}{R_6}, \tag{3.5}$$

still, the Inverting Integrator at $U4A$ will also define:

$$y_1 = -\frac{1}{C_2}\int \frac{c_1}{R_3} + \frac{x_1}{R_7} dt. \tag{3.6}$$

At the $AD633$ Multipliers, we can say that the output from $U2$ is:

$$W_{U2} = \frac{x_1^2}{10}, \tag{3.7}$$

and at $U1$, recognizing that $W_{U1} = c_1$ (from figure 3.2):

$$c_1 = \frac{b_1\left(a_1 - \frac{x_1^2}{10}\right)}{10}. \tag{3.8}$$

Deriving 3.4 we get:

$$\dot{x}_1 R_5 C_1 = -b_1. \tag{3.9}$$

From equation 3.5, and replacing in the previous equation:

$$\dot{x}_1 R_5 C_1 = y_1 \frac{R_6}{R_4} \tag{3.10}$$

and now replacing 3.8 in 3.6, and then the final y_1 in 3.10:

$$\dot{x}_1 R_5 C_1 = -\frac{R_6}{R_4 C_2} \int \frac{\frac{-\dot{x}_1 R_5 C_1 \left(a_1 - \frac{x_1^2}{10}\right)}{10}}{R_3} + \frac{x_1}{R_7} dt \tag{3.11}$$

Taking the derivative of equation 3.11, we get:

$$\ddot{x}_1 R_5 C_1 = -\frac{R_6}{R_4 C_2} \left(\frac{\frac{-\dot{x}_1 R_5 C_1 \left(a_1 - \frac{x_1^2}{10}\right)}{10}}{R_3} + \frac{x_1}{R_7} \right). \tag{3.12}$$

Rearranging the terms:

$$\ddot{x}_1 = -\frac{R_6}{R_4 R_5 C_1 C_2} \left(\frac{-\dot{x}_1 R_5 C_1 a_1}{10 R_3} + \frac{\dot{x}_1 R_5 C_1 x_1^2}{100 R_3} + \frac{x_1}{R_7} \right), \tag{3.13}$$

which is equal to:

$$\ddot{x}_1 = \frac{R_6}{R_4 C_2} \left(a_1 - \frac{x_1^2}{10} \right) \frac{\dot{x}_1}{10 R_3} - \frac{x_1}{R_5 C_1 R_7} \frac{R_6}{R_4 C_2} \tag{3.14}$$

Now we re-scale the differential equation by a factor α in order to eliminate the factor $\frac{1}{10}$ on the x_1^2 term. We do that by replacing $x_1 \to \gamma \tilde{x}_1$, so:

$$\gamma \ddot{\tilde{x}}_1 = \frac{R_6}{10 R_3 R_4 C_2} \left(a_1 - \frac{\gamma^2 \tilde{x}_1^2}{10} \right) \gamma \dot{\tilde{x}}_1 - \gamma \frac{R_6}{R_5 C_1 R_7 R_4 C_2} \tilde{x}_1, \tag{3.15}$$

and by making $\gamma^2 = 10$, we get:

$$\ddot{\tilde{x}}_1 = \frac{R_6}{10 R_3 R_4 C_2} \left(a_1 - \tilde{x}_1^2 \right) \dot{\tilde{x}}_1 - \frac{R_6}{R_5 C_1 R_7 R_4 C_2} \tilde{x}_1. \tag{3.16}$$

Now we apply again the same trick we presented in Appendix A, to transform this equation in (we change the notation back from $\tilde{x}_1$ to x_1 to a more usual perspective):

$$\ddot{x}_1 = \frac{a_1 R_6}{10 R_3 R_4 C_2} \left(1 - x_1^2 \right) \dot{x}_1 - \frac{R_6}{R_5 C_1 R_7 R_4 C_2} x_1. \tag{3.17}$$

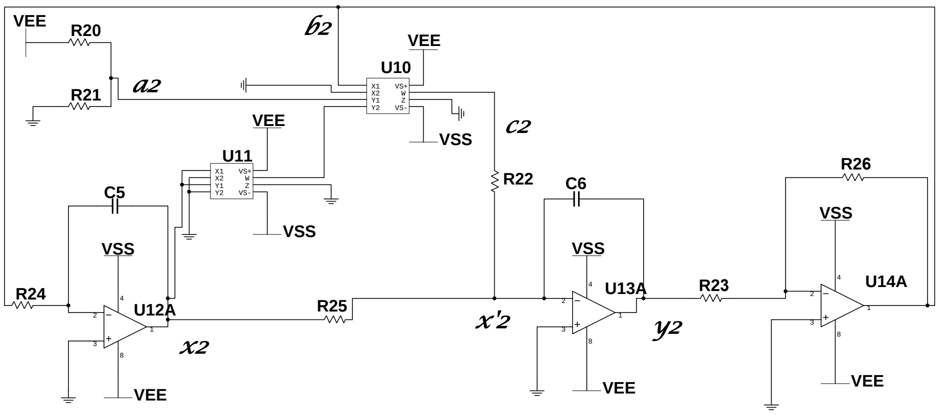


Figure 3.3: Circuit model of the vdP oscillator, representing the oscillator 1.

Defining:

$$\mu_1 = \frac{a_1 R_6}{10 R_3 R_4 C_2}, \tag{3.18}$$

and

$$\omega_1^2 = \frac{R_6}{R_5 C_1 R_7 R_4 C_2}, \tag{3.19}$$

we obtain:

$$\ddot{x}_1 + \mu_1 \left(x_1^2 - 1\right) \dot{x}_1 - \omega_1^2 x = 0, \tag{3.20}$$

which is exactly equation 1.1 - the vdP oscillator.

Considering now the second oscillator, represented in figure 3.3, we can easily derive the equation for its motion:

$$\ddot{x}_2 = \frac{a_2 R_{26}}{10 R_{22} R_{23} C_6} \left(1 - x_2^2\right) \dot{x}_2 - \frac{R_{26}}{R_{24} C_5 R_{25} R_{23} C_6} x_2, \tag{3.21}$$

in which we define the parameters $\mu_2 = \frac{a_2 R_{26}}{10 R_{22} R_{23} C_6}$ and $\omega_2^2 = \frac{R_{26}}{R_{24} C_5 R_{25} R_{23} C_6}$.

3.2 The Coupling Function

Following the argumentation we gave in section 2.1, in order to build a coupling function to the actual problem of two coupled oscillators that leads to the equation:

$$\ddot{x}_1 - \mu_1 (1 - x_1^2) \dot{x}_1 + \omega_1^2 x_1 + C_A (x_1 - x_2) + C_D (\dot{x}_1 - \dot{x}_2) = 0 \tag{3.22}$$

we will now develop the coupling circuit that will represent the terms in C_A and C_D. To do that we had to use circuit parts that could output the subtraction (figure 3.4a) and differentiation (figure 3.4b) terms, which have output functions, respectively (Appendix F) [1]:

$$y_{sub} = \frac{R_1 + R_2}{R_2}\frac{R_4}{R_3 + R_4}b - \frac{R_1}{R_2}a \tag{3.23}$$

$$y_{dif} = -RC\frac{da}{dt}. \tag{3.24}$$

In figure 3.5 we have the coupling scheme. From left to right, in figure 3.5, we have: oscillator 1) connects to R_{12} and oscillator 2) to R_{13}. Connected to $U6A$ is the Differential Amplifier, to $U7A$ and to $U9A$ are the Inverting Differentiators, and to $U8A$ is an Inverting Amplifier. So, for oscillator 1), the resulting equation will be the same as in the previous section, plus two terms resulting from the contact of resistors R_{18} and R_{19}, into x_1', in figure 3.2. Since $U4A$, in figure 3.2, is an Inverting Integrator, the additional terms from R_{18} and R_{19} will be summed inside the integral. We can easily see that the integral in $U4A$ will be:

$$\begin{aligned} y_1 = \quad & -\tfrac{1}{C_2}\int \tfrac{c_1}{R_3} + \tfrac{x_1}{R_7} + \\ & -\tfrac{1}{R_{19}}\tfrac{R_{14}}{R_{17}}\left(\tfrac{R_{16}}{R_{16}+R_{13}}\tfrac{R_8+R_{12}}{R_{12}}x_2 - \tfrac{R_8}{R_{12}}x_1\right) + \\ & -\tfrac{R_{11}C_3}{R_{18}}\left(\tfrac{R_{16}}{R_{16}+R_{13}}\tfrac{R_8+R_{12}}{R_{12}}\dot{x}_2 - \tfrac{R_8}{R_{12}}\dot{x}_1\right) dt \end{aligned} \tag{3.25}$$

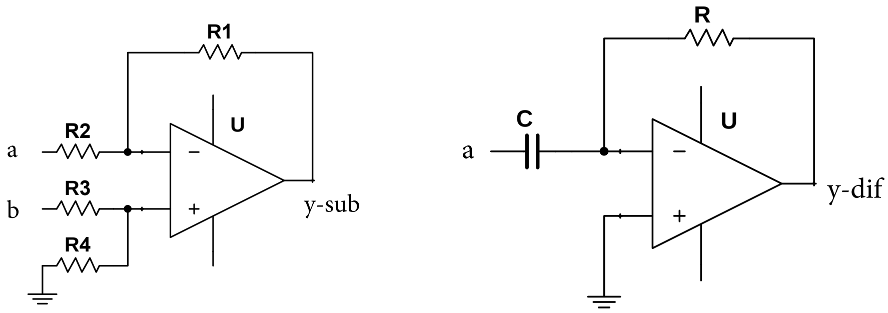
In order to get equations that resemble the equation of motion of the vdP oscillator, we have to make a simplification. The simplification is to consider that the Differential Amplifier in the coupling function has all resistors with the same value, i.e., we make $R_{16} = R_{13} = R_8 = R_{12}$. So the output expression is simply $y_{sub} = x_2 - x_1$. Therefore, we can say that for the example of the oscillator 1:

$$\begin{aligned} y_1 = \quad & -\tfrac{1}{C_2}\int \tfrac{c_1}{R_3} + \tfrac{x_1}{R_7} + \\ & -\tfrac{1}{R_{19}}\tfrac{R_{14}}{R_{17}}(x_2 - x_1) + \\ & -\tfrac{R_{11}C_3}{R_{18}}(\dot{x}_2 - \dot{x}_1)\, dt \end{aligned} \tag{3.26}$$

Substituting the last equation in the derivation for both oscillators in the previous section, we reach the resulting equation for the dynamics of the coupled system:

$$\begin{aligned} \ddot{x}_1 + \quad & \tfrac{a_1 R_6}{10R_3R_4C_2}\left(x_1^2 - 1\right)\dot{x}_1 + \tfrac{R_6}{R_5C_1R_4C_2R_7}x_1 + \\ & + \tfrac{R_6}{R_5C_1R_4C_2}\tfrac{1}{R_{19}}\tfrac{R_{14}}{R_{17}}(x_1 - x_2) + \tfrac{R_6}{R_5C_1R_4C_2}\tfrac{R_{11}C_3}{R_{18}}(\dot{x}_1 - \dot{x}_2) = 0 \end{aligned} \tag{3.27}$$

[1] Recall that the example figures have arbitrary reference numbers.



(a) Differential Amplifier **(b)** Inverting Differentiator

Figure 3.4: Parts of the Coupling Function

Recalling equations 2.11, 3.18 and 3.19, we can write:

$$\begin{aligned} C_{A1} &= \frac{R_6}{R_5C_1R_4C_2}\frac{1}{R_{19}}\frac{R_{14}}{R_{17}} \\ C_{D1} &= \frac{R_6}{R_5C_1R_4C_2}\frac{R_{11}C_3}{R_{18}} \end{aligned} . \tag{3.28}$$

Analogously for the second oscillator:

$$\begin{aligned} \ddot{x}_2 + &\frac{a_2R_{26}}{10R_{22}R_{23}C_6}\left(x_2^2-1\right)\dot{x}_2 + \frac{R_{26}}{R_{24}C_5R_{25}R_{23}C_6}x_2 + \\ &+\frac{R_{26}}{R_{24}C_5R_{23}C_6}\frac{1}{R_9}\left(x_2-x_1\right) + \frac{R_{26}}{R_{24}C_5R_{23}C_6}\frac{R_{14}}{R_{17}}\frac{R_{15}C_4}{R_{10}}\left(\dot{x}_2-\dot{x}_1\right) = 0, \end{aligned} \tag{3.29}$$

with:

$$\begin{aligned} C_{A2} &= \frac{R_{26}}{R_{24}C_5R_{23}C_6}\frac{1}{R_9} \\ C_{D2} &= \frac{R_{26}}{R_{24}C_5R_{23}C_6}\frac{R_{14}}{R_{17}}\frac{R_{15}C_4}{R_{10}} \end{aligned} . \tag{3.30}$$

3.2.1 Simplifying the Variables

Until now, we have as main variables the non-linearity control parameter, the frequency and the coupling terms, whose expressions are:

$$\mu_1 = \frac{a_1R_6}{10R_3R_4C_2} \tag{3.31}$$

$$\omega_1^2 = \frac{R_6}{R_5C_1R_7R_4C_2} \tag{3.32}$$

$$\mu_2 = \frac{a_2R_{26}}{10R_{22}R_{23}C_6} \tag{3.33}$$

$$\omega_2^2 = \frac{R_{26}}{R_{24}C_5R_{25}R_{23}C_6} \tag{3.34}$$

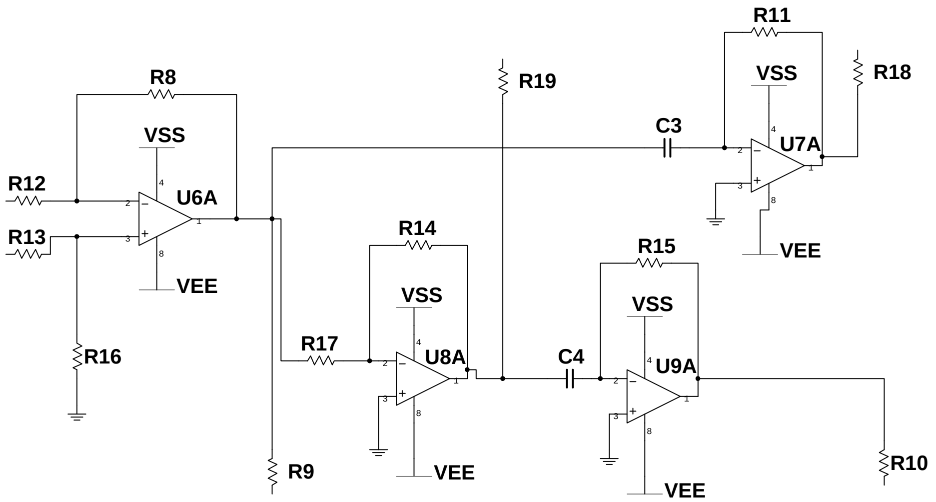


Figure 3.5: Coupling Scheme

$$C_{A1} = \frac{R_6}{R_5 C_1 R_4 C_2} \frac{1}{R_{19}} \frac{R_{14}}{R_{17}} \quad (3.35)$$

$$C_{D1} = \frac{R_6}{R_5 C_1 R_4 C_2} \frac{R_{11} C_3}{R_{18}} \quad (3.36)$$

$$C_{A2} = \frac{R_{26}}{R_{24} C_5 R_{23} C_6} \frac{1}{R_9} \quad (3.37)$$

$$C_{D2} = \frac{R_{26}}{R_{24} C_5 R_{23} C_6} \frac{R_{14}}{R_{17}} \frac{R_{15} C_4}{R_{10}} \quad (3.38)$$

Now we will simplify the definitions of these parameters (all parameters that we do not mention in these section, will keep their notation).

Non-linearity

We can define

$$R_3 = R_{22} = R_\mu, \quad (3.39)$$

since these are the resistors that will directly influence the non-linearity of the system.

Amplitude

The Inverting Amplifiers of each oscillator have a gain term represented by the fractions R_6/R_4 and R_{26}/R_{23}. We want to adjust the amplitude of the waves, in order to make the system near the amplitude of the limit cycle.

Since we just want to influence the phase, we can define:

$$\frac{R_6}{R_4} = \frac{R_{26}}{R_{23}} = \kappa, \tag{3.40}$$

where κ is a constant.

Coupling

As we said before, we want the system to be symmetric, so the corresponding coupling terms have to be the same. The resistors that control the coupling - R_9, R_{10}, R_{18} and R_{19} - have to be equal for each coupling type, for both oscillators. Hence,

$$R_9 = R_{19} = R_A, \tag{3.41}$$

and

$$R_{10} = R_{18} = R_D. \tag{3.42}$$

The resistors R_{11} and R_{15}, which belong to the Inverting Differentiators in the coupling function, have also to be the same, because they influence C_D:

$$R_{11} = R_{15} = R_{dif}. \tag{3.43}$$

The Inverting Amplifier, in the coupling function, will be defined with gain $g = 1$, so:

$$R_{14} = R_{17}. \tag{3.44}$$

Frequency

The resistors that influence the frequency of each oscillators will have the same values in each oscillator, meaning,

$$R_5 = R_7 = R_{x_1}, \tag{3.45}$$

and

$$R_{24} = R_{25} = R_{x_2}. \tag{3.46}$$

All the capacitors can have the same value: C.

Control Voltage

Considering the parameters a_1 and a_2, and noting that VEE is the input voltage (figure 3.2 and 3.3):

$$a_1 = \frac{R_2}{R_1 + R_2} VEE, \tag{3.47}$$

$$a_2 = \frac{R_{20}}{R_{21} + R_{20}} VEE, \tag{3.48}$$

we can define all resistors with the same value, resulting in:

$$a_1 = a_2 = \frac{VEE}{2} = a. \tag{3.49}$$

Result

These are the simplified terms:

$$\mu = \mu_1 = \mu_2 = \kappa \frac{a}{10 R_\mu C} \tag{3.50}$$

$$\omega_1^2 = \frac{k}{R_{x_1}^2 C^2} \tag{3.51}$$

$$\omega_2^2 = \frac{k}{R_{x_2}^2 C^2} \tag{3.52}$$

$$C_{A1} = \frac{k}{R_{x_1} C^2} \frac{1}{R_A} \tag{3.53}$$

$$C_{D1} = \kappa \frac{1}{R_{x_1} C} \frac{R_{dif}}{R_D} \tag{3.54}$$

$$C_{A2} = \frac{k}{R_{x_2} C^2} \frac{1}{R_A} \tag{3.55}$$

$$C_{D2} = \kappa \frac{1}{R_{x_2} C} \frac{R_{dif}}{R_D} \tag{3.56}$$

In the next chapter we will show the results for two coupled oscillators, by plotting their phase difference. Analogously, we analyse the experimental results for the same case, with pictures of the oscilloscope, and realise that the theoretical and experimental studies match. It is extremely important that, after analysing all theoretical conditions, the study has a practical application that confirms its results.

Chapter 4

On the Two Oscillators

Ubiquitously, from fundamental particles to major clusters of galaxies, synchronization results from cooperative interaction - an interaction between all types of objects. Since the sixteen hundreds, when Christiaan Huygens reported the synchronization between two pendula clocks hanging in the same wood wall, people have been very keen on the study of the dynamics of coupled units at all scales. Over the years, the number of reports of situations in nature that represented sync grew exponentially. Usually, the macroscopic phenomena involve a large number of units like the periodic blinking of fireflies, which is one of the most appreciated events. Yet, starting the study of synchronization by a finite large number of biological units is insanely hard. No field of science arose by one of its most complicated case. Thereafter, this study starts by its most fundamental case, which is the case of two oscillators with a simple coupling. Though simple, this case is very important, since the interaction of two lone units can be a first approximation to the interaction of every pair in an ensemble of any size - nearest neighbours interaction.

On this chapter we will recall the theoretical and experimental properties derived in the previous chapters, and present the corresponding results for the interaction of two coupled vdP oscillators connected by a simple coupling. This is a review chapter on what we have been working and it symbolises the basis for the following chapters, where the cases of more oscillators are considered.

4.1 Background

The study of two coupled oscillators is rather old. Nevertheless, it is the fundamental interaction theory of modern dynamics. With this work we want to show that a simple autonomous oscillator can be used in all branches (theory and practice) in agreement, and extrapolate to more than two units. Here, we present the final details for the study of two coupled van der Pol oscillators, with the theoretical and experimental results.

4.1.1 Scaling of Time

We have a coupling system that is govern by the equation:

$$\ddot{x}_i + \mu_i \left(x_i^2 - 1\right) \dot{x}_i + \omega_i^2 x_i + C_{Ai} \left(x_i - x_j\right) + C_{Di} \left(\dot{x}_i - \dot{x}_j\right) = 0, \tag{4.1}$$

with $i = 1, 2$, and $j = 1, 2 \neq i$. In our theoretical work, we have studied the dynamics of this system in the regime near $\omega \sim 1$. In the experimental study, such low frequencies can be a problem, considering that the majority of electronic parts do not work for extreme values of the frequency. So we have to scale this equation, in order to make the frequency near $\omega \sim 1000$ our baseline. So, we scale the time variable:

$$t \rightarrow s\tau, \tag{4.2}$$

where τ is the new time variable, and s is the scale factor. Equation 4.1 becomes:

$$\frac{1}{s^2}\ddot{x}_i + \mu_i \left(x_i^2 - 1\right) \frac{1}{s}\dot{x}_i + \omega_i^2 x_i + C_A \left(x_i - x_j\right) + \frac{1}{s} C_D \left(\dot{x}_i - \dot{x}_j\right) = 0. \tag{4.3}$$

Multiplying this equation by s^2:

$$\ddot{x}_i + s\mu_i \left(x_i^2 - 1\right) \dot{x}_i + s^2\omega_i^2 x_i + s^2 C_A \left(x_i - x_j\right) + sC_D \left(\dot{x}_i - \dot{x}_j\right) = 0, \tag{4.4}$$

which can be written as:

$$\ddot{x}_i + \tilde{\mu_i} \left(x_i^2 - 1\right) \dot{x}_i + \tilde{\omega}_i^2 x_i + \tilde{C_A} \left(x_i - x_j\right) + \tilde{C_D} \left(\dot{x}_i - \dot{x}_j\right) = 0. \tag{4.5}$$

Recalling the definition of our parameters, we to need to guarantee we can write the new ones as $(i = 1, 2)$:

$$\begin{aligned}
\tilde{\mu_i} &= s\mu_i = \tfrac{s\kappa a}{10 R_\mu C}, \\
\tilde{\omega}_i &= s\omega_i = \tfrac{sk}{R_{x_i} C}, \\
\tilde{C_{Ai}} &= s^2 C_{Ai} = \tfrac{s^2 k}{R_{x_i} C^2} \tfrac{1}{R_A}, \\
\tilde{C_D} &= sC_D = \tfrac{s\kappa}{R_{x_i} C} \tfrac{R_{dif}}{R_D}.
\end{aligned} \tag{4.6}$$

In order to make $\tilde{\omega} \sim 1000$, we need $s = 1000$.

Take into consideration that we have omitted any information about the parameter κ, because we consider it just a scaling parameter on the amplitude, and we will define it in the following sections.

4.1.2 Amplitude Dependence on the Parameter μ

Generically, a system like the vdP oscillator:

$$\ddot{x} - \mu\left(1 - x^2\right)\dot{x} + \omega^2 x = 0, \tag{4.7}$$

when treated perturbatively, has as a solution for the zeroth order (Appendix C) like:

$$x = 2\cos\left(t\right). \tag{4.8}$$

All other orders will depend on μ, and μ is small. Therefore, the limit cycle has its amplitude around 2.

With the change of variables $x \rightarrow \frac{x}{\sqrt{\mu}}$ we already used throughout this work, we get:

$$x = 2\sqrt{\mu}\cos\left(t\right), \tag{4.9}$$

which shows that the dependence of the amplitude on the parameter μ is proportional $\sqrt{\mu}$.

4.1.3 κ Parameter

With the zeroth order solution just showed, we see (before the change of variables) that the amplitude (for the zeroth order) is equal to 2. This will be the main contribution to the amplitude, so we can assume that other order will just influence perturbatively this value.

With the definition of μ we have reached:

$$\mu = \kappa\frac{a}{10R_\mu C}, \tag{4.10}$$

we can think of the gain κ as a constant of proportionality. So, we basically have:

$$\mu = \kappa\mu_0, \tag{4.11}$$

with μ_0 as the fundamental value - when gain is equal to unity ($\kappa = 1$). We are saying that we can define a parameter μ for the case $gain = 1$ in the Inverting Amplifier. However, we need gain to push the magnitude of the electronic oscillators to near the amplitude of the theoretical limit cycle, to avoid great transients or experimental problems.

As we have seen before, a small nonlinear parameter is mandatory, so it can be $\mu_0 = 0.05$ - quasi-harmonic solution. We also know that from equation 4.9:

$$A_0 = 2\sqrt{\mu_0}, \tag{4.12}$$

where A_0 is the fundamental amplitude. We know that we need to have the amplitude near the stable value of the limit cycle ($A = 2$), so we need to make sure that:

$$A = 2\sqrt{\mu}, \tag{4.13}$$

hence, we want a system with a gain κ, such that its nonlinear parameter is $\mu = 0.05$. Due to the definitions of A_0 and μ, we have:

$$A_0 = 2\sqrt{\frac{\mu}{\kappa}}, \tag{4.14}$$

so:

$$\sqrt{\frac{\mu}{\kappa}} = 1, \tag{4.15}$$

which results in:

$$\kappa = 0.05. \tag{4.16}$$

4.1.4 Attributing Values to the Parameters

As we have seen, $s = 1000$, $\mu_0 = 0.05$, and for the frequency $\omega_0 = 1$. We want to focus on the coupling in the derivatives, so we can choose a very small C_A ($C_A = 0.01$), and a larger value to C_D ($C_D = 0.20$). You may ask "why these values?". First, we have studied the case for $C_A = 0$, but in this particular experimental case it is impossible to mount the circuit for the coupling and make it physically zero - it would require an infinite resistor. Although we could remove the amplitude coupling part of the circuit, that would make the study less general - a simple change of the values of the components allows anyone interested to study different regimes. With the previous definitions, we can choose resistors and capacitors that will respect these properties. The choice will be always made to the fundamental case, i.e., when the gain ($\kappa = 1$), because it is the most simple case. The gain will influence, in the same way, all values, so it is not effectively important.

Frequency

For the parameter ω we used the unity, but now we scaled it by a factor of 1000:

$$\omega_0 = 1000. \tag{4.17}$$

Hence, for the case with no gain ($\kappa = 1$):

$$s\omega_i = \frac{1}{R_{x_i} C}$$

so we just have to choose any values. We chose $C = 10nF = 10^{-8}nF$ because it is a good, standard value for rapid oscillation. This results in:

$$R_{x_i} = 100k\Omega. \tag{4.18}$$

Non-linearity

For the parameter μ we have $\mu = s\kappa\mu_0$, but recalling conditions 4.11 and 4.15, we have $\mu = s$, so:

$$\mu = 1000 = \frac{a}{10R_\mu C}, \tag{4.19}$$

and because $a = 7.5$ ($VEE = 15V$, so a has the mathematical expression $a = \frac{VEE}{2} = 7.5$), we get the condition:

$$R_\mu = 75k\Omega. \tag{4.20}$$

Coupling

For the coupling term in the amplitudes, we have $\tilde{C}_{A0} = s^2 C_{A0}$, hence:

$$\tilde{C}_{A0} = 1000^2 \times 0.01 = \frac{1}{R_{x_i}C^2}\frac{1}{R_A}, \tag{4.21}$$

which results in:

$$R_A = 10M\Omega. \tag{4.22}$$

And for the coupling term in the derivatives, we do as we did for the non-linear control parameter. So, we have $\tilde{C}_{D0} = sC_{D0}$, with $C_{D0} = 0.20$:

$$\tilde{C_D} = 1000 \times 0.20 = \frac{1}{R_{x_i}C}\frac{R_{dif}}{R_D}, \tag{4.23}$$

which results in:

$$\frac{R_{dif}}{R_D} = 5 \tag{4.24}$$

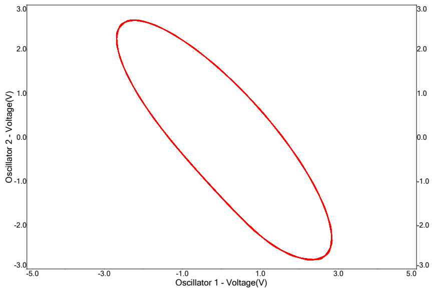
and defining $R_{dif} = 10k\Omega$, we get:

$$R_D = 50k\Omega. \tag{4.25}$$

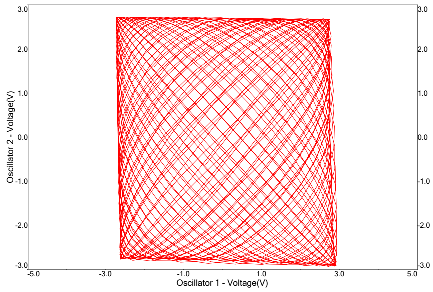
4.2 The Coupled Circuit

The full scheme of the circuit in the simulator representing the two oscillators and the coupling function can be seen in figure G.2 of Appendix G. A plot of the amplitude dependence, on the simulator, of both oscillators can be seen in figure 4.1a. From this figure we can clearly see that both oscillators have their motion in agreement. If the coupling were less intensive, i.e., changing R_{10} and R_{18} to $200k\Omega$, their motion would be independent, and the result is showed in figure 4.1b.

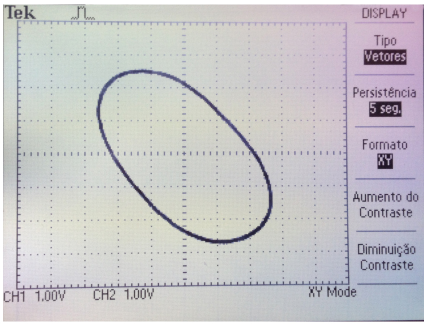
In table 4.1 we can see the true values of every component of the coupled system. The reason we chose to measure this is to show that an autonomous system has to be stable for the perturbations of the parameters (as discussed



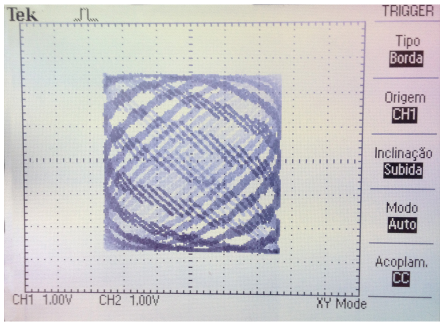
(a) Plot of the simulation showing the amplitude dependence of both oscillators.



(b) Plot of the simulation showing the lack of amplitude dependence of both oscillators.



(c) Oscilloscope picture of the amplitude dependence of x_1 and x_2.



(d) Oscilloscope picture of the lack of dependence of x_1 and x_2.

Figure 4.1: Comparison of simulation and experimental results

C	V	C	V	C	V	C	V
R1	98.98 $k\Omega$	R9	10.06 $M\Omega$	R17	99.09 $k\Omega$	R25	100.9 $k\Omega$
R2	99.84 $k\Omega$	R10	55.55 $k\Omega$	R18	55.50 $k\Omega$	R26	51.20 $k\Omega$
R3	75.05 $k\Omega$	R11	9.95 $k\Omega$	R19	10.17 $M\Omega$	C1	10.76 nF
R4	1.50 $k\Omega$	R12	99.20 $k\Omega$	R20	99.88 $k\Omega$	C2	10.28 nF
R5	98.84 $k\Omega$	R13	100.21 $k\Omega$	R21	99.51 $k\Omega$	C3	10.72 nF
R6	51.14 $k\Omega$	R14	99.76 $k\Omega$	R22	75.08 $k\Omega$	C4	10.24 nF
R7	98.60 $k\Omega$	R15	9.94 $k\Omega$	R23	1.50 $k\Omega$	C5	10.72 nF
R8	98.84 $k\Omega$	R16	99.05 $k\Omega$	R24	101.0 $k\Omega$	C6	10.31 nF

Table 4.1: Measured values (V) of the components (C) of the system of two coupled oscillators.

through the thesis). It is only important to choose the parameters in a certain range (small error). And how do we know that? We set up the corresponding circuit, and plugged in the oscilloscope to see figure 4.1c. We clearly see the dependence, just similar to what we observed with the simulator in figure 4.1a. Analogously, in figure 4.1d we verify that the motion flows independently, just identical to the one in figure 4.1b.

4.3 Mathematical Enforcement

In Chapter 2 we have develop methods to attain equations for the phase difference. An integration of those equations, specifically 2.37 for the KB method, and also the four equations that are composed by 2.44 and 2.45, allow us to analyse if both analytical and numerical studies are in concordance. Nonetheless, a introductory point on the definition of the phase will be done, since we need to clarify its definition in all approaches.

4.3.1 Definition of the Phase

In order to compare both solutions, we need first to define the phase in both methods. So we will compare the KB and the numerical method (N). Hence:

$$\begin{aligned} KB \quad & \begin{cases} x = & A\cos\left(\omega_s t + \varphi\right) \\ \dot{x} = & -A\omega_s\left(\omega_s t + \varphi\right) \end{cases} \\ N \quad & \begin{cases} x = & \mathcal{A}\cos\left(\phi\right) \\ \dot{x} = & \mathcal{A}\sin\left(\phi\right) \end{cases} \end{aligned} \tag{4.26}$$

We want to plot the phase difference in the usual way as a plot of the polar variables, such that:

$$\begin{aligned} \mathcal{A}\cos\left(\phi\right) &= A\cos\left(\omega_s t + \varphi\right) \\ \mathcal{A}\sin\left(\phi\right) &= -A\omega_s\left(\omega_s t + \varphi\right) \end{aligned} \tag{4.27}$$

Solving this system, we verify that:

$$\begin{aligned} \mathcal{A} = & \quad \sqrt{A^2\cos^2\left(\omega_s t + \varphi\right) + A^2\omega_s^2\sin^2\left(\omega_s t + \varphi\right)} \\ \phi = & \quad \arctan\left(-\omega_s \tfrac{\sin(\omega_s t + \varphi)}{\cos(\omega_s t + \varphi)}\right) \end{aligned} \tag{4.28}$$

Making $\omega_s = 1$:

$$\begin{aligned} \mathcal{A} = & \quad A \\ \phi = & \quad \arctan\left(\frac{\sin(-t-\varphi)}{\cos(-t-\varphi)}\right) \end{aligned} \tag{4.29}$$

resulting in:

$$\begin{aligned} \mathcal{A} = & \quad A \\ \phi = & \quad -t-\varphi \end{aligned} \tag{4.30}$$

One has to be very careful every time *phase* is defined in a new approach. The main problem will be when plotting the phase differences. The methods will end in distinct results for the same problem. In the methods herein studied, and as demonstrated now, we have to plot the phase differences as (ϕ - numerical, φ - KB):

$$\phi_2 - \phi_1 = \varphi_1 - \varphi_2. \tag{4.31}$$

4.4 Plot of the Solutions

An integration of the equations and a plot of the phase difference is here presented, paying attention to the condition 4.31, and using the following values ($i = 1, 2$ means both values are equal):

$$\begin{aligned} \mu_i &= 0.05 \\ \omega_1 &= 1.01 \\ \omega_2 &= 0.99 \\ C_{Ai} &= 0.01 \\ C_{Di} &= 0.2 \end{aligned} \tag{4.32}$$

C4 - Numerical and KB plot.nb

The resulting plot in figure 4.2 show us that $KBAM$ give us the average of the numerical method, which ensure us the adequateness of the model. We can recognize that the oscillators synchronize for the above mentioned values, with a phase difference around 0.1.

In the next chapter we repeat the same approach, although considering the case of three oscillators. Plots of the solutions and pictures of the experimental parts are enclosed, and a brief discussion of the differences between the cases of two and three coupled oscillators.

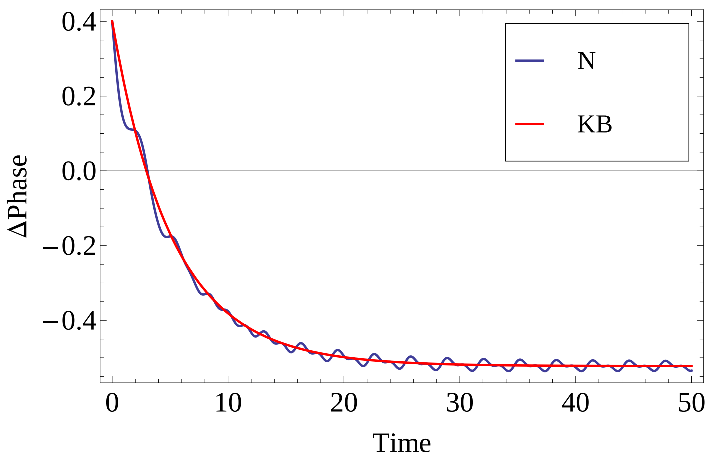


Figure 4.2: Plot of the solutions representing the phase difference, using numerical N and KB methods.

Chapter 5

On the Three Oscillators

Reality itself is ambiguous - one cannot say if something really exists or not with one hundred percent certainty. All we can do is observe, study, think, and try to understand what is going on. So, what we call reality, is something that we cannot prove that it is real. What people, and in particular scientists, try to do is to improve the notion of what is observed (directly or not), and approaching what someone courageous might call *the truth*. Of course the truth with 100% certainty does not exist due to the flawed mathematics we have nowadays, however, it is not necessary. We can approach this enigmatic truth with a really tiny error. Here, we are trying to convince the reader that a dynamical world is surrounding us and everything interacts with everything (or almost everything, so a particle physicists would say), and its interaction results, many times, in synchronization.

In order to study an ensemble with a reasonable number of elements, we now focus on the three elements case, which is a very important middle step. In this and the next chapter we will show that the study of two oscillators is the base for any number of oscillators interacting in a ensemble, and that sync appears with few conditions. Every couple of units interact, independently that if they interact with others. The main idea is that every interaction happen between two units at a time, i.e., the coupling of three units is the sum of 3 pairs of two coupled units: 1+2, 2+3, and 1+3.

The following pages present our work on three coupled oscillators. The case of three oscillators is, obviously, equal for the NN interaction and the all-to-all interaction, just like in the two oscillators case (more on the different kinds of interaction, and the correspondent terms, on Chapter 6).

5.1 One More

The study of the coupling of three oscillators is based on the pairwise interaction discussed so far in this work. The reason we choose to shed light on

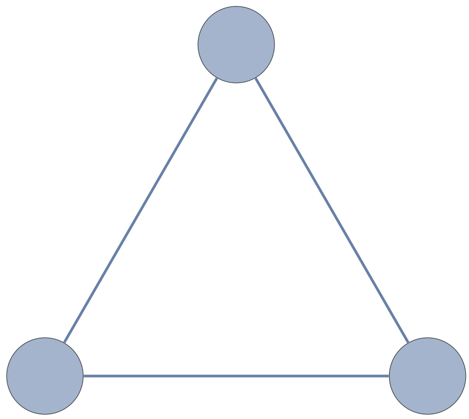


Figure 5.1: Graph with Three Simply Connected Nones

this specific case, and not any other case, is discussed in Chapter 6.

Here, we redo the study of previous chapters in a neater and briefer way, not emphasizing the middle calculations as we did so far. As we can see in figure 5.1, we have three nodes in a graph, representing three oscillators, and three links between them representing the simple coupling, meaning that between every pair in the system there is the above mentioned pair-wise interaction. The system is simple, and because of that we expect that every unit influence the other two with in the same way, i.e., the influence of oscillator 1 on oscillator 2 has the same magnitude of 1 on 3, and the same happens to any permutation of $\{1, 2, 3\}$ in this example. Of course we are not concerned with *distances* and other variables, what would make the interactions in this system look different. Here we are only interested in knowing the dynamics of a system formed by three coupled vdP oscillators.

5.1.1 Equation of Motion

As we did in Chapter 2, we need to calculate some mathematical tools for this case. The graph for the system represented in figure 5.1 defines the following Kirchhoff matrix:

C5 - Coupling Term 3vdp.nb

$$K_{3vdp} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}. \tag{5.1}$$

The output functions will not change from the two oscillators case (and again, they will not change either for any number of oscillators - see Chapter 6), because we want the same internal properties, of each oscillator, to participate in the coupling. In this way, the output functions are:

$$H_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \tag{5.2}$$

$$H_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}. \tag{5.3}$$

The vector of the internal variables is:

$$\vec{x} = \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ x_3 \\ \dot{x}_3 \end{pmatrix} \tag{5.4}$$

and so, the coupling function (recall from 2.4) is:

$$\begin{bmatrix} 0 \\ C_A\left(-2x_1+x_2+x_3\right)+C_D\left(-2\dot{x}_1+\dot{x}_2+\dot{x}_3\right) \\ 0 \\ C_A\left(-2x_2+x_1+x_3\right)+C_D\left(-2\dot{x}_2+\dot{x}_1+\dot{x}_3\right) \\ 0 \\ C_A\left(-2x_3+x_2+x_1\right)+C_D\left(-2\dot{x}_3+\dot{x}_2+\dot{x}_1\right) \end{bmatrix} \tag{5.5}$$

and the consequent equations of motion (recall 1.11):

$$\begin{aligned} &\ddot{x}_1+\mu_1(x_1^2-1)\dot{x}_1+\omega_1^2x_1-C_A\left(-2x_1+x_2+x_3\right)-C_D\left(-2\dot{x}_1+\dot{x}_2+\dot{x}_3\right)=0 \\ &\ddot{x}_2+\mu_2(x_2^2-1)\dot{x}_2+\omega_2^2x_2-C_A\left(-2x_2+x_1+x_3\right)-C_D\left(-2\dot{x}_2+\dot{x}_1+\dot{x}_3\right)=0 \\ &\ddot{x}_3+\mu_3(x_3^2-1)\dot{x}_3+\omega_3^2x_3-C_A\left(-2x_3+x_2+x_1\right)-C_D\left(-2\dot{x}_3+\dot{x}_2+\dot{x}_1\right)=0 \end{aligned} \tag{5.6}$$

It is interesting to notice that the interaction of three units is equal to the sum of the interaction of two pairs of units, i.e., $x_1 - x_2$ and $x_1 - x_3$, for example. In Chapter 6 we explain the interests in this new coupling term. For now, we are only interest in deriving the numerical and approximated equations for the phase, and plotting them to see if we keep the concordance.

5.1.2 Krylov-Bogoliubov Averaging Method

The KB model for the case of more than two oscillators lacks the need of emphasizing its relevance. Since it works for two oscillators case, and the only thing we are changing is the number of oscillators. The equations for each pair, in an ensemble, remain the same, so the method must work. Nevertheless, here we write the resulting approximated equations for three oscillators. Recalling the derivation of the method in Chapter 2, and now with coupling 5.5, we get three amplitude equations:

$$\begin{aligned} \dot{A}_1 \; &= \tfrac{\mu_1}{2}A_1-\tfrac{\mu_1}{8}A_1^3+\tfrac{C_A}{2\omega_s}\left(A_2\sin\left(\varphi_2-\varphi_1\right)+A_3\sin\left(\varphi_3-\varphi_1\right)\right)+ \\ &+\tfrac{C_D}{2}\left(A_2\cos\left(\varphi_2-\varphi_1\right)-2A_1+A_3\cos\left(\varphi_3-\varphi_1\right)\right) \end{aligned} \tag{5.7}$$

$$\begin{aligned}\dot{A}_2 &= \tfrac{\mu_1}{2}A_2 - \tfrac{\mu_1}{8}A_2^3 + \tfrac{C_A}{2\omega_s}\left(A_3\sin\left(\varphi_3-\varphi_2\right) + A_1\sin\left(\varphi_1-\varphi_2\right)\right) + \\ &+ \tfrac{C_D}{2}\left(A_3\cos\left(\varphi_3-\varphi_2\right) - 2A_2 + A_1\cos\left(\varphi_1-\varphi_2\right)\right)\end{aligned} \tag{5.8}$$

$$\begin{aligned}\dot{A}_3 &= \tfrac{\mu_1}{2}A_3 - \tfrac{\mu_1}{8}A_3^3 + \tfrac{C_A}{2\omega_s}\left(A_1\sin\left(\varphi_1-\varphi_3\right) + A_2\sin\left(\varphi_2-\varphi_3\right)\right) + \\ &+ \tfrac{C_D}{2}\left(A_1\cos\left(\varphi_1-\varphi_3\right) - 2A_3 + A_2\cos\left(\varphi_2-\varphi_3\right)\right)\end{aligned} \tag{5.9}$$

and three phase equations:

$$\begin{aligned}\dot{\varphi}_1 &= \tfrac{\omega_1^2-\omega_s^2}{2\omega_s} - \tfrac{C_A}{2\omega_s}\left(A_2\cos\left(\varphi_2-\varphi_1\right) - 2A_1 + A_3\cos\left(\varphi_3-\varphi_1\right)\right) + \\ &+ \tfrac{C_D}{2}\left(A_2\sin\left(\varphi_2-\varphi_1\right) + A_3\sin\left(\varphi_3-\varphi_1\right)\right)\end{aligned} \tag{5.10}$$

$$\begin{aligned}\dot{\varphi}_2 &= \tfrac{\omega_2^2-\omega_s^2}{2\omega_s} - \tfrac{C_A}{2\omega_s}\left(A_3\cos\left(\varphi_3-\varphi_2\right) - 2A_2 + A_1\cos\left(\varphi_1-\varphi_2\right)\right) + \\ &+ \tfrac{C_D}{2}\left(A_3\sin\left(\varphi_3-\varphi_2\right) + A_1\sin\left(\varphi_1-\varphi_2\right)\right)\end{aligned} \tag{5.11}$$

$$\begin{aligned}\dot{\varphi}_3 &= \tfrac{\omega_3^2-\omega_s^2}{2\omega_s} - \tfrac{C_A}{2\omega_s}\left(A_1\cos\left(\varphi_1-\varphi_3\right) - 2A_3 + A_2\cos\left(\varphi_2-\varphi_3\right)\right) + \\ &+ \tfrac{C_D}{2}\left(A_1\sin\left(\varphi_1-\varphi_3\right) + A_2\sin\left(\varphi_2-\varphi_3\right)\right)\end{aligned} \tag{5.12}$$

Here, we decided to integrate with random variables, i.e., a random number generator, let us call it *rand(low,top)*, which result is a random real number between *low* and *top*. All other variables take previous definitions:

$$\begin{aligned} \omega_i &\to rand\left(0.95, 1.05\right) & \mu_i &= 0.05 \\ A_i\left(0\right) &\to rand\left(1.7, 2.3\right) & C_D &= 0.2 \\ \varphi_i\left(0\right) &\to rand\left(0, 2\pi\right) & C_A &= 0.01 \\ \omega_s &= 1 \end{aligned} \tag{5.13}$$

C5 - KB Integration 3vdp.nb

An example of integration with this method and these values can be seen in figure 5.2. A deeper discussion on random variables is provided in Chapter 6.

5.1.3 Numerical Study

The numerical study of this case follows the straightforward idea we mentioned in Chapter 2. We use polar coordinates, to change everything in system 5.6 and after a pile of calculations (that any software can do for you)

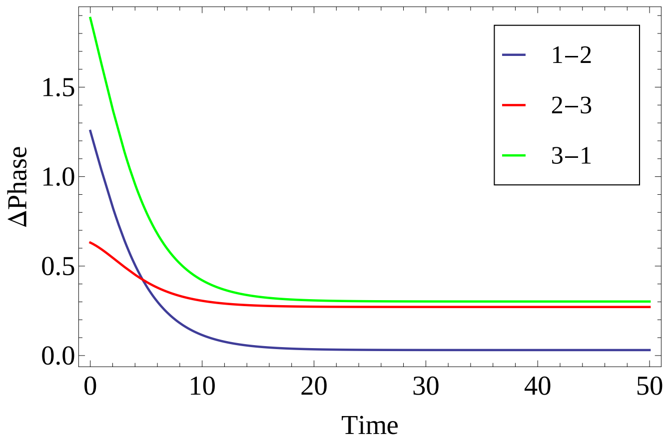


Figure 5.2: Plot of a solution for three coupled oscillators with KB method

we reach a system of 6 equations: 3 for each amplitude and 3 for each phase. Here we do not show the resulting equations (you can see similar and more general equations in Chapter 6). For now we will restrain ourselves to the plot of three numerically integrated coupled oscillators - figure 5.3 - and the correspondent equivalence with the KB method - figure 5.4.

C5 - N Integration.nb

C5 - Numerical Integration 3vdp.nb

5.2 The circuit

In Chapter 3 we obtained the vdP circuit through the correspondent equation, and the corresponding coupling function for two oscillators. Although it should be clear that the coupling of two units is the fundamental one, and the base for all forthcoming studies, we also have seen that the mathematical expressions are different in the coupling functions for three oscillators. So the circuit structure for the coupling part will also be different. The main concern, once again, is to represent, as reliable as possible, the coupling function as circuit parts, while using the same structure for the oscillator.

5.2.1 Coupling Function

As we have already seen, the coupling function for three oscillators results from the two units interaction. The result is weighted terms, meaning that for each oscillator, its own amplitude will count two times, due to the subtracting influence of the other two oscillators.

To reproduce this function, we will need a Non-inverting weighted summer - figure 5.5 - a Non-Inverting Amplifier - figure 5.6 - and some other parts we already used in Chapter 3 and Appendix F. The idea is to use the Non-Inverting Amplifier to produce the double weight in each oscillator, and

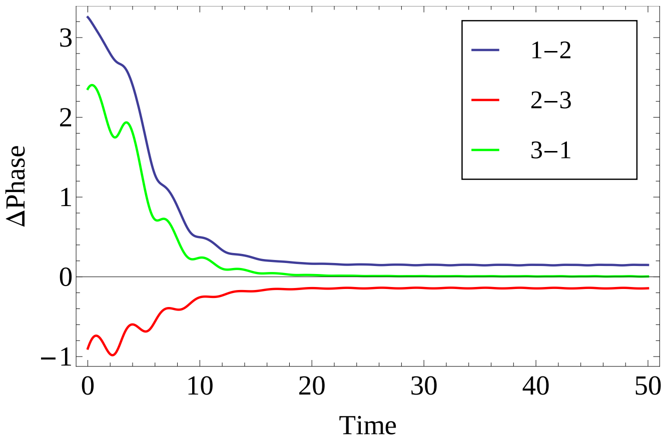


Figure 5.3: Numerical integration of three coupled vdP oscillators

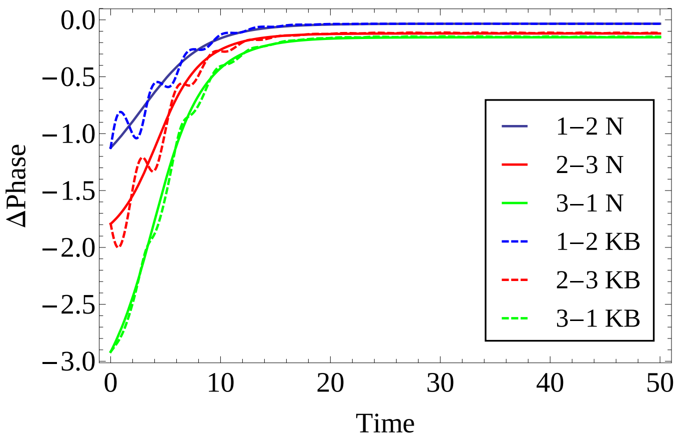


Figure 5.4: Comparison between KG and numerical integration of three coupled vdP oscillators

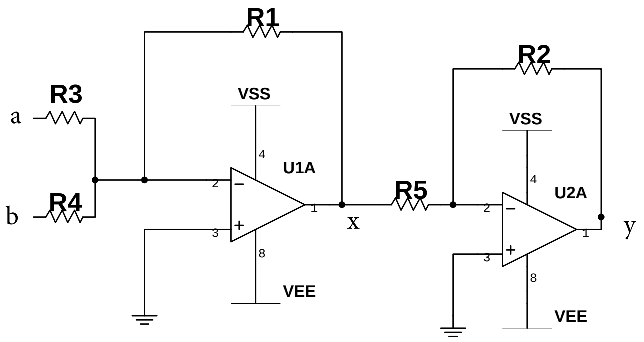


Figure 5.5: Non-Inverting Weighted Summer

then, with a Differential Amplifier and a Non-Inverting Weighted Summer, subtract the sum of the other two oscillators, creating the desired term, e.g., $(-x_1 + 2x_2 - x_3)$.

In figure 5.7 we can clearly see an example of what happens in our model of three coupled circuits. As inputs we have x_1, x_2 and x_3, and this is for oscillator 1 (the one double weighted in figure 5.7). In the left part, the voltage (i.e., amplitude) of oscillator 1 is doubled, and enters in the part on the right. The Non-Inverting Weighted Summer sums oscillators 2 and 3, and the Differential Amplifier produces the different between $2x_1$ and $x_2 + x_3$. Then we just need a Inverting Differentiator for the coupling term referent to C_D.

5.2.2 Results and Discussion

In the final part of this chapter we show the results, in a parallel of the simulational, the experimental and the theoretical parts. We used the same reference values and parameters as in the case for two oscillators. One of the main concerns of our work was connecting the different approaches to the problem. In the previous chapters we already demonstrated the link between the numerical and the approximated KB theoretical methods, and the link between the simulation and the experimental results. Here we show the link between theory and practice, by plotting the numerical and simulations results, with the pictures from the oscilloscope (5.8, **??** and **??**). We accomplished our goal in each triplet of figures, realising the agreement of all parts. Although the agreement may seem vague, we have to consider that the software that we used to integrate the equations use numerical methods, and even with the computational errors, it still has a high resolution and the solutions are symmetric and pretty. While, the software that integrated the circuit, uses highly sophisticated methods to approach the numerical

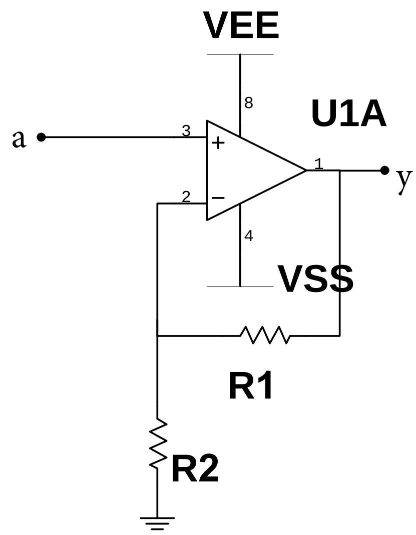


Figure 5.6: Non-Inverting Amplifier

outputs of each component. Hence the solution is not exactly the same as in the theory. The output functions we have been considering are rough approximations, which influence the results considerably. The circuit itself and all the errors associated with it (like measurement or manufacturing errors), change the desired output to something less symmetric.

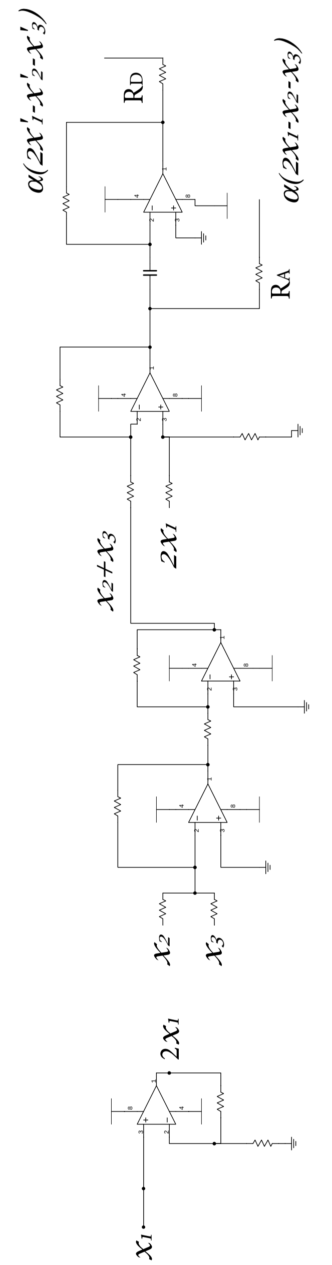
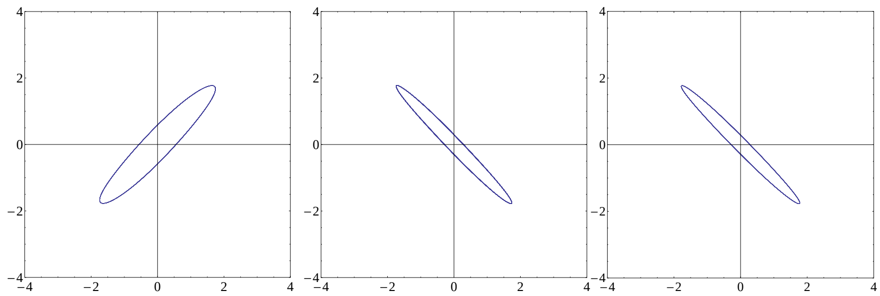
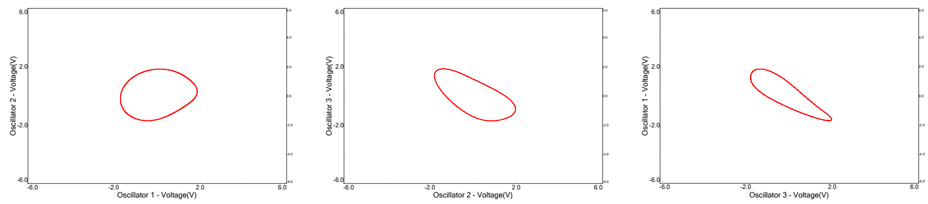


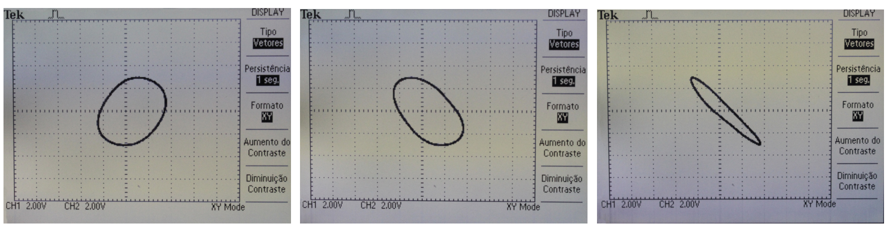
Figure 5.7: Scheme of the coupling function needed for each oscillator in the coupling of three units. The parameter α is present just to remember that the function that outputs the coupling scheme has some parameters of proportionality, due to the values of the resistors and capacitors.



(a) Numerical (b) Numerical (c) Numerical



(d) Simulator (e) Simulation (f) Simulation



(g) Oscilloscope (h) Oscilloscope (i) Oscilloscope

Figure 5.8: Comparison of the different approaches for the three coupled oscillators sorted by interacting pairs, i.e., 1 and 2 (column on the left), 2 ans 3 (column in the middle) and 3 and 1 (column on the right).

Chapter 6

A Lot of Oscillators

Ernest Hemingway wrote in his letters *"A man's got to take a lot of punishment to write a really funny book."* [11]. A man who did write a really funny, and also educational, book, is one of the greatest leaders of synchronization study, Steven Strogatz. Concerning the beauty and complexity of nature, he wrote *"Most of everyday life is nonlinear, and the principle of superposition fails spectacularly. If you listen to your two favorite songs at the same time, you won't get double pleasure!"* [12]. We are not trying to approach the genius of Hemingway or Strogatz, however, this thesis has brought a lot of positive punishment, and we are trying to show you that this spectacular nonlinear world is full of rhythm and pace. Every oscillating thing might synchronize with other oscillating things in its environment. The sufficient conditions to attain a synchronize state are very weak, as we have learnt from this work, and considering the great amount of synchronized motion we observe, at all scales in nature, we can definitely say it is a master law of this dynamical universe.

The present work tries to deliver an idea to the common non-scientific reader, and a well-fundamented notion to the scientist, that we can jump to any case of synchronization one can imagine and still find sync. In this chapter, we will explore an idea on how to make the dynamical approach more general, increasing the number of oscillators to "the hard cases" category. We already mentioned that extreme number of oscillators (too little or too many), are simple and old cases. Our main focus here is the case for ten coupled van der Pol oscillators, reaching the main goal: obtaining positive results in a more complex and general case, starting from the real basic example.

6.1 The Interaction

In a ensemble of oscillating constituents, the main part is trying to understand how they are correlated, which is the correlation term, how to measure the interaction, and all questions concerning how different units interact. The global dynamics of the system can be understood from the properties of each individual. So far, we have neglected the concept of distance, and we will keep the main study that way, so we can consider that every pair has the same mutual interaction - same *distance*. This is called *nearest neighbours* interaction. As an alternative, we could have what is called *all-to-all* interaction. Up to three units case, both approaches are the same. But while we add more units, the approaches start to diverge. For example, for a 10 units case, the nearest neighbours interaction is drawn in figure 6.1a and the all-to-all interaction is in figure 6.1b. As another example, the Kirchhoff matrices[1] for these cases are:

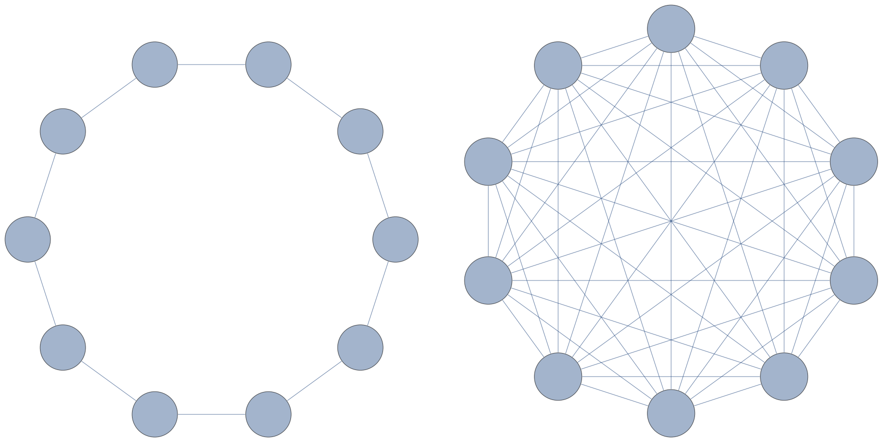
C6 - Coupling Term 10vdp - nn.nb
C6 - Coupling Term 10vdp - all-to-all.nb

$$K^{NN}_{10vdp} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \tag{6.1}$$

$$K^{all}_{10vdp} = \begin{bmatrix} 9 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 9 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 9 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 9 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 9 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 9 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 9 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 9 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 9 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 9 \end{bmatrix} \tag{6.2}$$

All-to-all interaction will not be considered since it is not obvious that a system of n units can perform this kind of coupling. It would mean that, in a huge system, the interaction strength would influence the units with instantaneous speed. In a more classical case of pairwise interaction, the influence of a unit propagates through the other units.

[1] In appendix, you can find two more examples of coupling terms for an all-to-all coupling: *C6 - Coupling Term 20vdp - all-to-all.nb*, and *C6 - Coupling Term 30vdp - all-to-all.nb*



(a) Ten oscillators with nearest neighbours interaction

(b) Ten oscillators all-to-all interaction

Figure 6.1: Graphs with ten nodes each

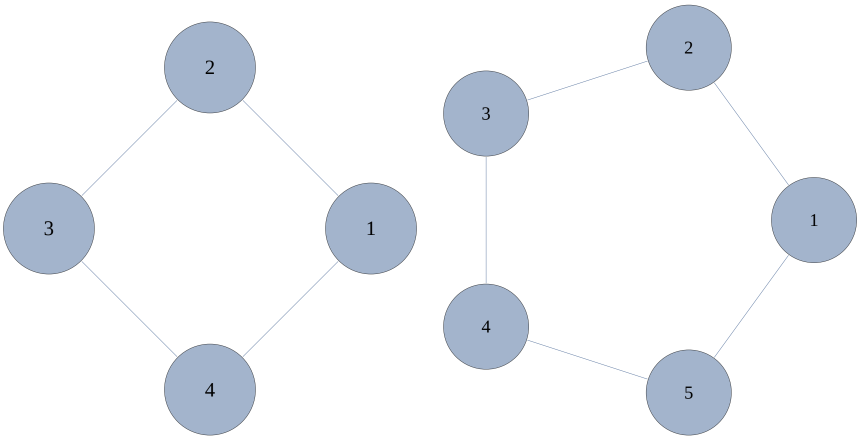
6.1.1 Nearest Neighbours Interaction

The coupling terms in a scenario with more oscillators will follow the case of three oscillators studied in Chapter 5. Since we are dealing with a one dimensional problem - the ring. In a two dimensional problem each unit has 4 nearest neighbours, and in a three dimensional problem, 6 nearest neighbours.

Imagine we have four coupled units. The coupling term for one unit will be the pairwise interaction with its nearest neighbours, hence, it will be, for unit 3, for example, $C\left(-2x_3 + x_4 + x_2\right)$ (C is just representing any general coupling). As we can see in figure 6.2a, unit 1 it is not a nearest neighbour of unit 3. The same thing happens in the case for 5 units in figure 6.2b. If we consider node 5, the nearest neighbours terms are x_4 and x_1 and the coupling term is $C\left(-2x_5 + x_1 + x_4\right)$. This is how we can formalize an idea for the interaction terms for any number of units in a graph, or in a more physical way of saying, any number of oscillators in a dynamical system, by understanding the three units case.

In order to derive analytical equations for the amplitude and the phase, we make use of all ideas above. From the point of view of nearest neighbours, and as we mentioned in this chapter, we can say that the equation of motion for the oscillator i is:

$$\begin{split}\ddot{x}_i + \mu_i(x_i^2 - 1)\dot{x}_i + \omega_i^2 x_i - \\ - \left(C_A\right)_i\left(-2x_i + x_{i+1} + x_{i-1}\right) - \left(C_D\right)_i\left(-2\dot{x}_i + \dot{x}_{i+1} + \dot{x}_{i-1}\right) = 0,\end{split} \tag{6.3}$$



(a) Graph with four nodes **(b)** Graph with 5 nodes

Figure 6.2: Two different graphs with 4 and 5 nodes respectively, to illustrate which nearest neighbours terms will affect each unit

with periodic boundary conditions, i.e., for N units u, $u_{N+1} \to u_1$. We have to write this system in the form:

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= -\mu_i(x_i^2 - 1)\dot{x}_i - \omega_i^2 x_i + \\ &\quad + (C_A)_i\,(-2x_i + x_{i+1} + x_{i-1}) + (C_D)_i\,(-2\dot{x}_i + \dot{x}_{i+1} + \dot{x}_{i-1}) \end{aligned} \tag{6.4}$$

Again, we use polar coordinates transformation:

$$\begin{aligned} x_i\,(t) &= A_i \cos\left(\phi_i\right) \\ y_i\,(t) &= A_i \sin\left(\phi_i\right) \end{aligned} \tag{6.5}$$

and also:

$$\begin{aligned} \dot{x}_i\,(t) &= \dot{A}_i \cos\left(\phi_i\right) - A\dot{\phi}_i \sin\left(\phi_i\right) \\ \dot{y}_i\,(t) &= \dot{A}_i \sin\left(\phi_i\right) + A\dot{\phi}_i \cos\left(\phi_i\right) \end{aligned} \tag{6.6}$$

giving us:

$$\begin{aligned} \dot{A}_i &= \cos\left(\phi_i\right)\dot{x}_i + \sin\left(\phi_1\right)\dot{y}_i \\ \dot{\phi}_i &= \tfrac{1}{A_i}\left(\cos\left(\phi_i\right)\dot{y}_i - \sin\left(\phi_i\right)\dot{x}_i\right) \end{aligned} \tag{6.7}$$

which enable us to obtain, after a good amount of calculations, the general equations for the amplitude and the phase, by substituting 6.4 in 6.7, resulting in:

$$\begin{aligned}
\dot{A}_i &= \sin(\phi_i)\left[A_i\left(1-2\left(C_A\right)_i-\omega_i^2\right)\cos(\phi_i)+\left(\mu_i-2C_{Di}\right)\sin(\phi_i)+\right.\\
&\quad +A_{i+1}\left(\left(C_A\right)_i\cos(\phi_{i+1})+\left(C_D\right)_i\sin(\phi_{i+1})\right)+\\
&\quad \left.+A_{i-1}\left(\left(C_A\right)_i\cos(\phi_{i-1})+\left(C_D\right)_i\sin(\phi_{i-1})\right)\right]
\end{aligned} \tag{6.8}$$

$$\begin{aligned}
\dot{\phi}_i &= \mu_i\cos(\phi_i)\sin(\phi_i)\left(1-A_i^2\cos^2(\phi_i)\right)-\left(2\left(C_A\right)_i+\omega_i^2\right)\cos^2(\phi_i)-\\
&\quad -\sin^2(\phi_i)-\left(C_D\right)_i\sin(2\phi_i)+\\
&\quad +\tfrac{A_{i+1}}{A_i}\cos(\phi_i)\left(\left(C_A\right)_i\cos(\phi_{i+1})+\left(C_D\right)_i\sin(\phi_{i+1})\right)+\\
&\quad +\tfrac{A_{i-1}}{A_i}\cos(\phi_i)\left(\left(C_A\right)_i\cos(\phi_{i-1})+\left(C_D\right)_i\sin(\phi_{i-1})\right)
\end{aligned} \tag{6.9}$$

These are the general analytical equations for the nearest neighbours interaction of an ensemble of any quantity of vdP oscillators.

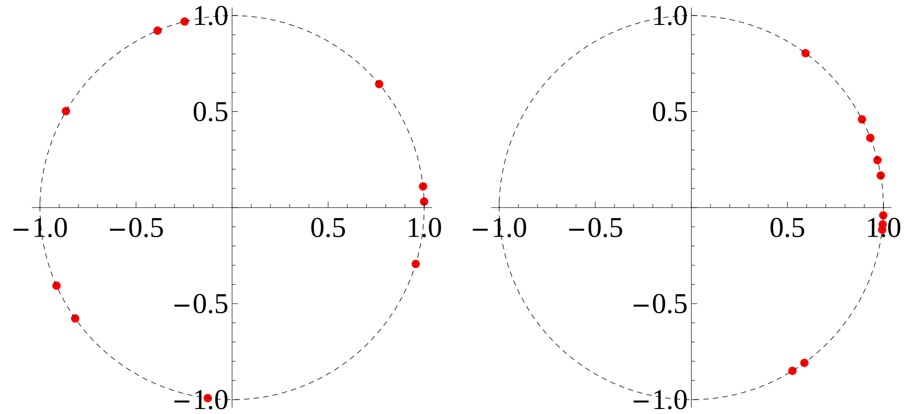
6.2 Integration

Today, the main problems focus on the cases where the number of units are more than a few and less than too many, since these special cases have trivial approximations with easy solutions, solved long ago.

In the following examples, and throughout the remaining sections, we will consider the case where all parameters characterizing the units are allowed to vary and take random values:

$$\begin{array}{rlrl}
\omega_i & \to rand\,(0.95, 1.05) & \mu_i & \to rand\,(0.04, 0.06)\\
A_i\,(0) & \to rand\,(1.7, 2.3) & \left(C_D\right)_i & \to rand\,(0.18, 0.22)\\
\varphi_i\,(0) & \to rand\,(0, 2\pi) & \left(C_A\right)_i & \to rand\,(0.005, 0.015)\\
A_i\,(0) & \to rand\,(1.9, 2.1) & \phi_i\,(0) & \to rand\,(0, 2\pi)
\end{array} \tag{6.10}$$

Here we would like to make a note on the variation of the parameters. Allowing all parameters to change makes the problem more realist, such that in a group of people one cannot say that there are n number of humans and make all humans alike. Although the core is similar, it is not the same. So, these variations account for the heterogeneity of the system of “identical” oscillators. Like in the case of fireflies, we deal with real biological living creatures,



(a) $t = 1$ **(b)** $t = 500$

Figure 6.3: Plot on the unit circle of the phase difference between every pair of oscillators in a ring of ten oscillators

not mathematical objects we can assimilate as oscillators with fixed frequencies and amplitudes. Nevertheless, we cannot say these variations are reasonable for a system of random individuals (i.e., humans, fireflies, etc.), since in a random system there might be one, or more, individuals that are tremendously different from the average, which are not considered here. We consider small variations to the normal values.

Now, we show the results from the integration of the equations in a manner that is easy for the reader to analyse.

In figure 6.1a we already showed the graph for ten oscillators with NN interaction on a ring. To show the phenomenon of phase synchronization we plot, in the unit circle, the phase differences between consecutive units, $\phi_{i+1} - \phi_i$, with the conditions shown in 6.10. We show the configurations at time $t = 1$, and some time after the rapid transient, at time $t = 500$, in figure . It is interesting to notice the random distribution of initial phase differences, and then the result of the coupling, making the phase difference range narrower. We also need to say, that the final condition achieved in figure 6.3 is stable, and it will remain in that configuration, what is a clear sign of a synchronous behaviour.

The cases for more then ten oscillators are also interesting, still, the study for ten is already enough to the goals of this work. We wanted to pursue and motivate the study for a considerable number of oscillators, beginning from the most simple case of two coupled oscillators. Here, a long way from the simplest case, we can clearly see the influence and the importance of the initial study. From the experimental study of two and three oscillators, with the model here presented, we know that if a experimental study was done with ten oscillators, or more, it could be guided by this procedure, and would get something very similar to these results. Also, the case for more oscillators needs more rigorous conditions, i.e., in order to achieve global synchronization, more specific properties are needed, which is somewhat logic - if we have a system with more units, we need to be more careful in how to work them to perform a synchronize behaviour.

6.3 Frequency Vector

Throughout this work we have seen the importance of the phase, and the unimportance of the amplitude. Now, we will dedicate this section to the study of the importance of the frequency vector. The frequency vector is a vector which elements correspond to the natural frequency of each oscillator, i.e., the one parameter that represents the individuals. It is like a identity number for the oscillator. Of course, if it identifies the oscillators, it is important for us to know how can this frequency vector change, and that is what we intend here.

6.3.1 Range

We have been working with a frequency range of $\omega_i \in [0.95, 1.05]$. There is a clear notion that the narrower the distribution of frequencies, the easier the system will obtain a synchronized motion, even for small values of the coupling strength. We decided to analyse this parameter, and vary it, to determine what changes with the variation of vector ω. Basically, we will spread the frequency range, and with the same random number generator tool, generate a wider frequency range vector, and repeat the calculations.

C6 - Range.nb

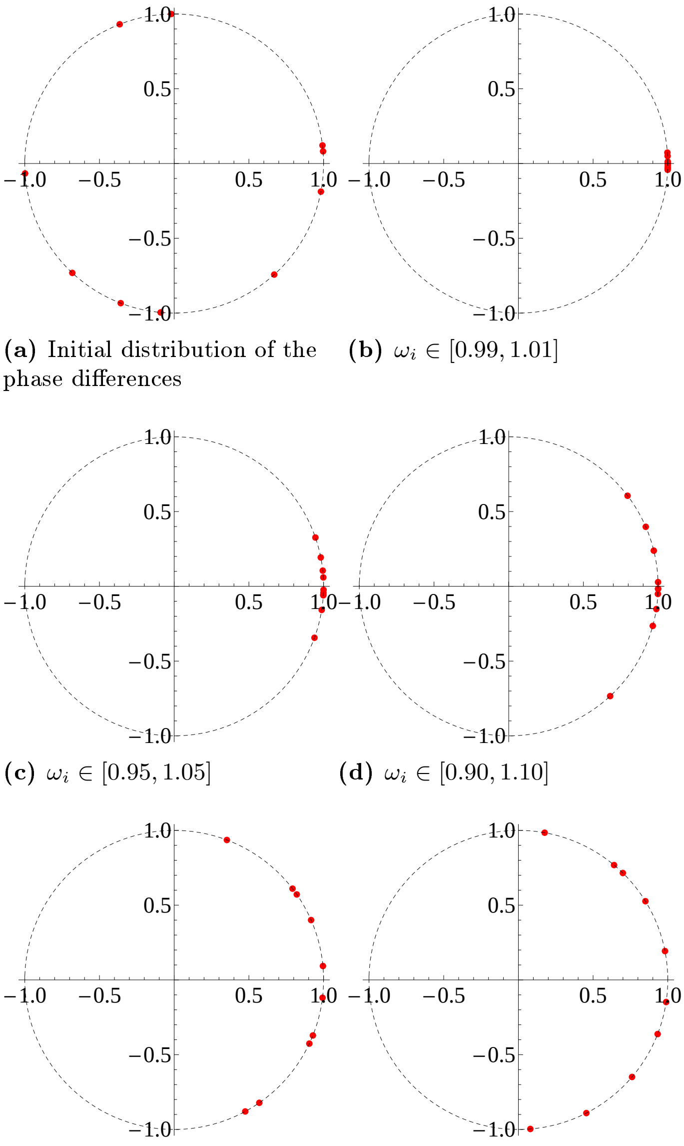
We will only change the frequency vector, and keep the values of the other parameters. Plots for different vectors at $t = 500$ are shown in figure 6.4.

6.3.2 Order

Another point of view that it is interesting to study is the order of the units in the system, in this case, in the ring. For three different frequency vectors, we did the following numerical procedure:

C6 - Order.nb

- Randomly choose a frequency distribution vector;
- Draw the plot corresponding to the final configuration of the phase differences;
- Shuffle one hundred times the order of the frequency elements in the vector, and calculated the phase differences to each resulting randomly shuffled frequency vector;



(a) Initial distribution of the phase differences

(b) $\omega_i \in [0.99, 1.01]$

(c) $\omega_i \in [0.95, 1.05]$

(d) $\omega_i \in [0.90, 1.10]$

(e) $\omega_i \in [0.85, 1.15]$

(f) $\omega_i \in [0.80, 1.20]$

Figure 6.4: Plot of the final configuration (at $t = 500$) of the phase differences from $\omega_i \in [0.99, 1.01]$ to $\omega_i \in [0.8, 1.2]$.

- Choose the frequency vector with the lowest phase differences amplitude;

- Plot the corresponding phase difference diagram in the unit circle, to confirm that a different configuration have been reached.

- Repeat the procedure for different frequency vectors.

We decided to evaluate three different regimes:

- $\omega \in [0.85, 1.15]$;

- $\omega \in [0.90, 1.10]$;

- $\omega \in [0.95, 1.05]$;

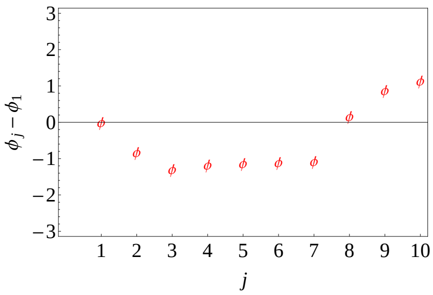
The results are summarised in figures 6.5, 6.6 and 6.7 respectively. In these sets of figures, elements a) and b) represent the frequency distributions: the random distribution (column on the left) and the optimised distribution (column on the right); c) and d) represent the sequent phase difference from oscillator j to oscillator 1; and e) and f) represent the distribution of phase differences in the unit circle.

6.4 Closure

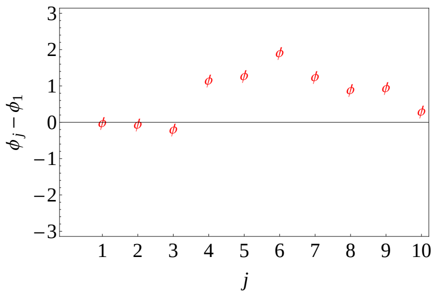
Dear reader, this is the end of our work and I really do wish you have enjoyed it. We had a wonderful walk-through the majesty of nature in one of its astonishing behaviours - the synchronization. We hope we could led you from the early developments on synchronize motion by analysing the simplest case - two coupled van der Pol oscillators. Our intentions became higher, and as we aimed for more, we rapidly climbed through three oscillators, to literally any number of them. We understood some of their properties, and we realised how stable this state is.

Of course, we have not only found greatness. We had a lot of hard work, and a lot of wrong ways before finding the right one, but we did. In between, we also gain acknowledgement on the limitations of the methods. To study units with large non-linearity, it would require a deep change in the process. Nevertheless, all the main ideas are present in this work, and we can confidently say that any other work, more or less complex, can be done from this one, or with parallel ideas.

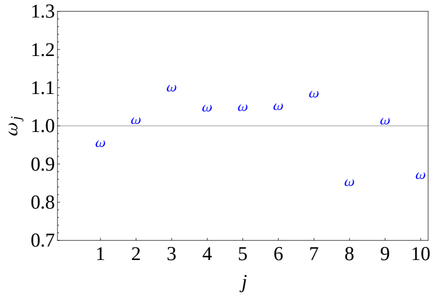
Synchronization is, in my opinion, a really amazing phenomenon. Just imagine the wonderful things that can be built with an arbitrary number of units, which goal is to make a macro effect. Even if the units are small, like fireflies, hundreds of them will light up the sky and embellish your night. And that is why *nature* itself is the best justification to study nature. You will never be bored.



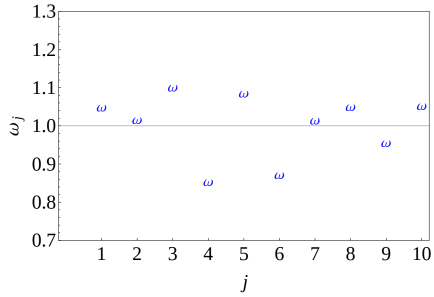
(a) Phase difference to oscillator 1 (R)



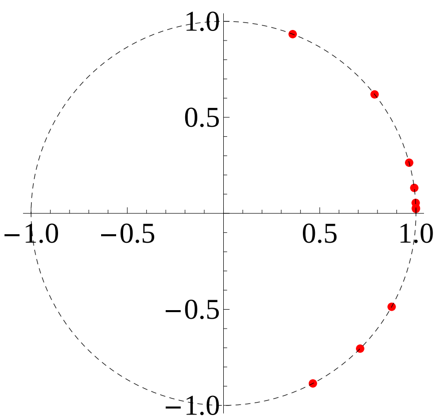
(b) Phase difference to oscillator 1 (O)



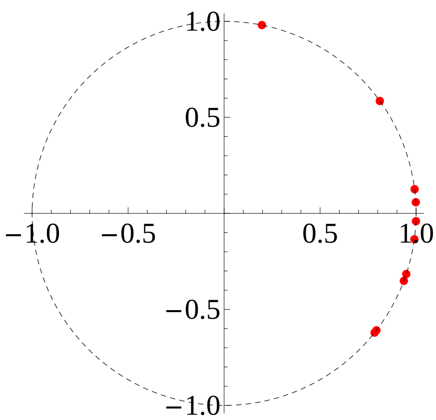
(c) Frequency distribution (R)



(d) Frequency distribution (O)

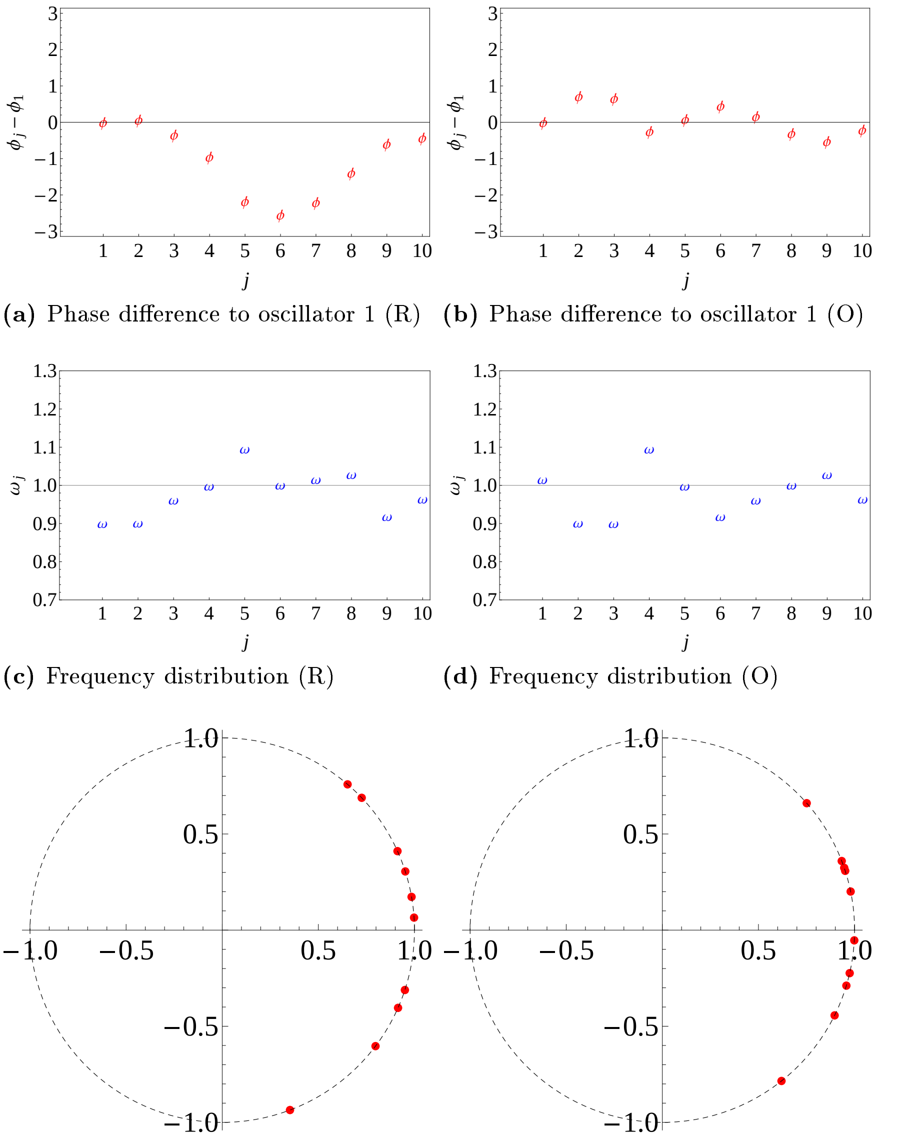


(e) Phase differences (R)



(f) Phase differences (O)

Figure 6.5: Plots with $\omega \in [0.85, 1.15]$ random (R) and optimised (O)

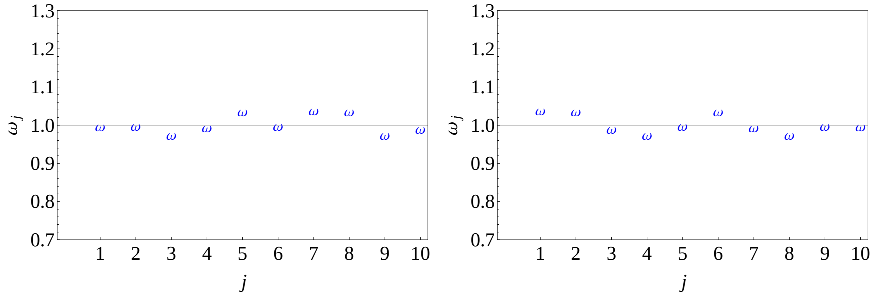


(a) Phase difference to oscillator 1 (R) **(b)** Phase difference to oscillator 1 (O)

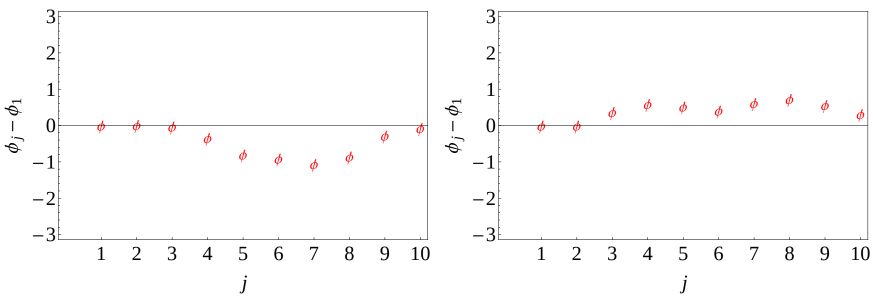
(c) Frequency distribution (R) **(d)** Frequency distribution (O)

(e) Phase differences (R) **(f)** Phase differences (O)

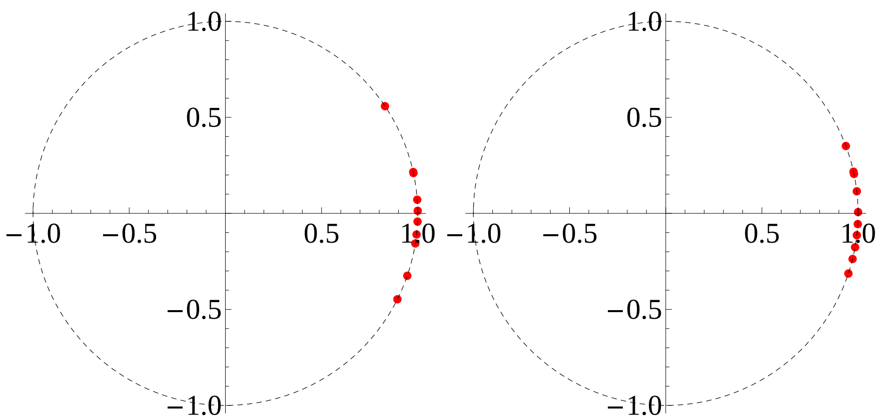
Figure 6.6: Plots with $\omega \in [0.90, 1.10]$ random (R) and optimised (O)



(a) Phase difference to oscillator 1 (R) **(b)** Phase difference to oscillator 1 (O)



(c) Frequency distribution (R) **(d)** Frequency distribution (O)



(e) Phase differences (R) **(f)** Phase differences (O)

Figure 6.7: Plots with $\omega \in [0.95, 1.05]$ random (R) and optimised (O)

Appendix A

Equivalence Between van der Pol Equations

Here we will demonstrate the equivalence between the vdP equations, although the critical points of each have different values. The vdP equation can be written as:

$$x''(t) + (x^2(t) - \mu)x'(t) + \omega^2 x(t) = 0 \tag{A.1}$$

and, we can manipulate it as:

$$\begin{aligned}
&x''(t) + (x^2(t) - \mu)x'(t) + \omega x(t) = \\
&= x''(t) + \mu(\tfrac{x^2(t)}{\mu} - 1)x'(t) + \omega^2 x(t) \\
&= \mu\left(\tfrac{1}{\mu}x''(t) + (\tfrac{x^2(t)}{\mu} - 1)x'(t) + \tfrac{\omega^2}{\mu}x(t)\right) \\
&= \mu\left(\tfrac{1}{\mu}x''(t) + (\left[\tfrac{x(t)}{\sqrt{\mu}}\right]^2 - 1)x'(t) + \omega^2\tfrac{x(t)}{\mu}\right) \\
&= \mu\tfrac{\sqrt{\mu}}{\sqrt{\mu}}\left(\tfrac{1}{\mu}x''(t) + (\left[\tfrac{x(t)}{\sqrt{\mu}}\right]^2 - 1)x'(t) + \omega^2\tfrac{x(t)}{\mu}\right) \\
&= \tfrac{\mu}{\sqrt{\mu}}\left(\tfrac{\sqrt{\mu}}{\mu}x''(t) + (\left[\tfrac{x(t)}{\sqrt{\mu}}\right]^2 - 1)\sqrt{\mu}x'(t) + \sqrt{\mu}\omega^2\tfrac{x(t)}{\mu}\right) \\
&= \tfrac{\mu}{\sqrt{\mu}}\left(\tfrac{1}{\sqrt{\mu}}x''(t) + (\left[\tfrac{x(t)}{\sqrt{\mu}}\right]^2 - 1)\sqrt{\mu}x'(t) + \omega^2\tfrac{x(t)}{\sqrt{\mu}}\right)
\end{aligned}$$

Due to of equation A.1:

$$\frac{\mu}{\sqrt{\mu}}\left(\frac{1}{\sqrt{\mu}}x''(t) + (\left[\frac{x(t)}{\sqrt{\mu}}\right]^2 - 1)\sqrt{\mu}x'(t) + \omega^2\frac{x(t)}{\sqrt{\mu}}\right) = 0 \tag{A.2}$$

Defining a new term:

$$y(t) = \frac{1}{\sqrt{\mu}} x(t), \tag{A.3}$$

we have:

$$y''(t) + \mu(y^2(t) - 1)y'(t) + \omega^2 y(t) = 0. \tag{A.4}$$

In this last equation, the non-linearity is modelled by the parameter μ, so when $\mu = 0$, the equation represents the harmonic oscillator. The same does not happen for the first equation, which has the critical point for $\mu = x^2$. The mapping of the two forms, A.1 and A.4 of the vdP equation, does not carry over to $\mu = 0$.

Appendix B

Limit Cycle of the van der Pol Oscillator

In order to justify the study of synchronization with van der Pol oscillators, we have to show they perform self-sustained oscillations. This implies that a limit cycle will appear in its phase space. To do that, we use Liénard's theorem.

Liénard's theorem is a theorem that proofs that a system in a certain form has one stable limit cycle. It applies to systems in the form:

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0. \tag{B.1}$$

Defining:

$$F(x) := \int_0^x f(\zeta)\, d\zeta, \tag{B.2}$$

we can write what is called a Liénard system:

$$\begin{cases} \dot{x} &= y - F(x) \\ \\ \dot{y} &= -g(x) \end{cases}. \tag{B.3}$$

This theorem has simple premisses, however its proof is not trivial. The proof lies outside the core of this work and can be seen, for example, in reference [13].

If:

- $F, g \in \mathcal{C}^1$
- F, g odd functions of x;
- $xg(x) > 0$ for $x \neq 0$;
- $F(0) = 0$;

- $\dot{F}(0) < 0$;
- F has one root at $x = a$;
- and F is monotonically increasing for $x \geq a$;

then, system B.3 has exactly one limit cycle, and it is stable. If we recall the vdP oscillator:

$$\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + \omega^2 x = 0, \tag{B.4}$$

we observe that we can write it as a Liénard system:

$$\begin{cases} \dot{x} & = y - \mu\left(\frac{x^3}{3} - x\right) \\ \\ \dot{y} & = -x \end{cases} \tag{B.5}$$

So, $F(x) = \mu\left(\frac{x^3}{3} - x\right)$ and $g(x) = x$, do verify all of Liénard's theorem conditions, meaning that the vdP oscillator has an unique stable limit cycle. Also, we see that $(\dot{x}, \dot{y}) = (0, 0)$, hence the search for fixed points is only satisfied for $(x, y) = (0, 0)$.

Appendix C

Perturbation Theory

Perturbation theory is widely used in all branches of science. The complexity of many subjects requires the usage of such topic, and, for the sake of knowledge, it does work acceptably well. The main idea is linked to the advantage of certain domain regions, i.e., we do not need to know every state of the universe, since the big bang, to study how the sun will explode - we can concentrate in a narrow range of time. In perturbation theory, one simply chooses a particular interval of some parameter, and study the simplified system. In specific, here we will derive an approximated analytical solution for the van der Pol oscillator in the small non-linearity range ($\mu \ll 1$). Of course it will not work for other values of μ, but those values lie outside of our interests.

Many studies on perturbation theory have been made over the years, and many different ways of studying systems have been used. Here, we will use a generic method - for more examples and methods, see, for example, reference [8]. We start by recalling the equation of the vdP oscillator:

$$\ddot{x} + \mu \left(x^2 - 1\right) \dot{x} + x = 0. \tag{C.1}$$

The core of perturbation theory sets on the following idea: we can choose a solution x that is defined by a series on the small parameter, in this case μ. Hence:

$$x = x_0 + \mu x_1 + ... \tag{C.2}$$

where x_0 is the zeroth order solution, x_1 is the first order solution, and so on. Now we substitute x in the equation of motion, and we collect the terms with the same order in μ, and at the same time we eliminate all terms with $\mu^n : n \geq 2$. The result for μ^0 is:

$$\ddot{x}_0 + x_0 = 0, \tag{C.3}$$

and for μ^1:

$$\ddot{x}_1 + x_1 = -\left(x_0^2 - 1\right) \dot{x}_0. \tag{C.4}$$

We know because of what we showed in Appendix B, that the vdP oscillator has a stable limit cycle. So, as initial conditions, we assume that for the zeroth order on the amplitude it will reach a stable value a, hence, we can choose this value as:

$$x_0(0) = a. \tag{C.5}$$

We have only reduced the transient - there is no problem at all in choosing the *destination* point as the initial condition. The other initial conditions are zero ($x_1(0)$, $\dot{x}_0(0)$ and $\dot{x}_1(0)$). We can now solve this problem. The solution for μ^0 equation is:

$$x_0(t) = a\cos(t), \tag{C.6}$$

and using this solution in C.4 we have:

$$\ddot{x}_1 + x_1 = -\left(a^2\cos^2(t) - 1\right) a\sin(t). \tag{C.7}$$

Using the trigonometric relation:

$$\cos^2(t)\sin(t) = \frac{1}{4}\left(\sin(t) + \sin(3t)\right), \tag{C.8}$$

we write:

$$\ddot{x}_1 + x_1 = a\sin(t)\left(1 - \frac{a^2}{4}\right) - \frac{a^3}{4}\sin(3t). \tag{C.9}$$

The term proportional to $\sin(t)$ will resonate with the zeroth order solution (proportional to $\cos(t)$) - secular term. We need to avoid this resonance and eliminate this term. We do that by choosing $a = 2$, which leave us with:

$$\ddot{x}_1 + x_1 = -2\sin(3t). \tag{C.10}$$

Solving this equation and adding it in the solution, we get:

$$x = 2\cos(t) - \mu\sin^3(t).$$

We can compare this approximated solution with the numerical solution in figure C.1.

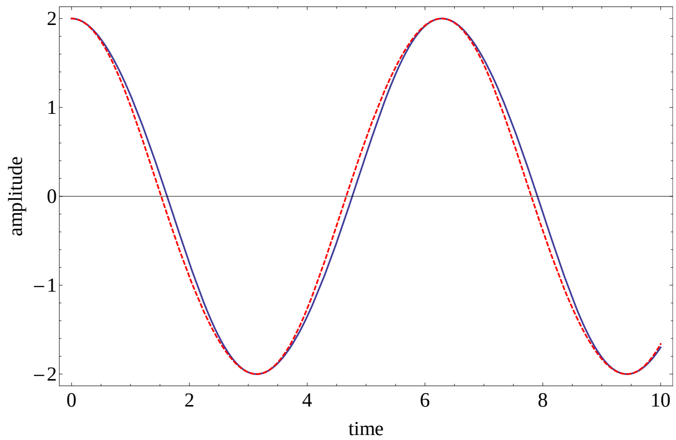


Figure C.1: Numerical (blue, thick line) and perturbative (purple, dashed line) solutions of the vdP oscillator for $\omega = 1$ and $\mu = 0.05$.

We also wanted to show that for a small parameters μ the zeroth order solution is $x = 2\cos(t)$, which means that the amplitude is stable near the magnitude of 2.

Appendix D

Fourier Coefficients

The general form of the Fourier coefficients, with variables of amplitude A and a generic phase $\widetilde{\varphi} = \omega t + \varphi$. are:

$$K_0(A) = \frac{1}{2\pi}\int_0^{2\pi} f\left(x, \dot{x}\right) \cos\left(\widetilde{\varphi}\right) d\widetilde{\varphi}, \tag{D.1}$$

$$P_0(A) = \frac{1}{2\pi}\int_0^{2\pi} f\left(x, \dot{x}\right) \sin\left(\widetilde{\varphi}\right) d\widetilde{\varphi}, \tag{D.2}$$

$$K_n(A) = \frac{1}{\pi}\int_0^{2\pi} f\left(x, \dot{x}\right) \cos\left(\widetilde{\varphi}\right) \cos\left(n\widetilde{\varphi}\right) d\widetilde{\varphi}, \tag{D.3}$$

$$L_n(A) = \frac{1}{\pi}\int_0^{2\pi} f\left(x, \dot{x}\right) \cos\left(\widetilde{\varphi}\right) \sin\left(n\widetilde{\varphi}\right) d\widetilde{\varphi}, \tag{D.4}$$

$$P_n(A) = \frac{1}{\pi}\int_0^{2\pi} f\left(x, \dot{x}\right) \sin\left(\widetilde{\varphi}\right) \cos\left(n\widetilde{\varphi}\right) d\widetilde{\varphi}, \tag{D.5}$$

$$Q_n(A) = \frac{1}{\pi}\int_0^{2\pi} f\left(x, \dot{x}\right) \sin\left(\widetilde{\varphi}\right) \sin\left(n\widetilde{\varphi}\right) d\widetilde{\varphi}, \tag{D.6}$$

recalling that:

$$f\left(x, \dot{x}\right) = f\left(A\cos\left(\widetilde{\varphi}\right), -A\omega \sin\left(\widetilde{\varphi}\right)\right). \tag{D.7}$$

Appendix E

Derivation of the Approximate Equation of Motion

In Chapter 2 we say that in equation 2.21, and using equation 2.18, we can eliminate the term 2). So, recalling equation 2.18:

$$\dot{A}\cos(\omega_s t+\varphi) - A\dot{\varphi}\sin(\omega_s t+\varphi) = 0, \tag{E.1}$$

which is the condition for the first derivative to be similar to the harmonic solutions. We also used the coordinate transformation:

$$\begin{aligned} a &= Ae^{i\varphi} \\ a^* &= Ae^{-i\varphi} \end{aligned}, \tag{E.2}$$

so if we write the harmonic solution for $\dot{x}$, i.e., $\dot{x} = -A\omega_s \sin(\omega_s t+\varphi)$, we get:

$$\begin{aligned} \dot{x} &= -\omega_s A\left(\frac{e^{i(\omega_s t+\varphi)} - e^{-i(\omega_s t+\varphi)}}{2i}\right) \\ &= \frac{i\omega_s}{2}\left(a\,e^{i\omega_s t} - a^* e^{-i\omega_s t}\right) \end{aligned}. \tag{E.3}$$

Nevertheless, if we perform the derivative of:

$$x = A\cos(\omega_s t+\varphi) = a\,e^{i\omega_s t} + a^* e^{-i\omega_s t}, \tag{E.4}$$

we get:

$$\dot{x} = \dot{a}\,e^{i\omega_s t} + i\omega_s a\,e^{i\omega_s t} + \dot{a}^* e^{-i\omega_s t} - i\omega_s a^* e^{-i\omega_s t}, \tag{E.5}$$

hence the terms we need to eliminate are:

$$\dot{a}^* e^{-i\omega_s t} + \dot{a}\,e^{i\omega_s t} = 0, \tag{E.6}$$

proving that the term 2) in 2.21 is zero.

Appendix F

Derivation of the Output Expressions of Electronic Parts

In Chapter 3 we discuss the circuit for two coupled oscillators, and in particular, the coupling of the system, using some devices which have specific output functions that are of interest to us. Here we will be deriving these output functions. Everywhere needed, we will call the tension on the (+) and (-) inputs of the operational amplifiers as u_+ and u_-.

Inverting Integrator

In figure 3.1a we have the scheme of an Inverting Integrator. The current that flows through the resistor is the same as the one through the capacitor, so:

$$\frac{a}{R} = C\frac{dy_{int}}{dt}. \tag{F.1}$$

So:

$$y_{int} = -\frac{1}{C}\int \frac{a}{R}dt. \tag{F.2}$$

Inverting Amplifier

In figure 3.1c, because $u_- = 0$, and the current in R_1 is the same as in R_2, with a minus sign, we simply have:

$$\frac{a}{R_1} = -\frac{y_{inv}}{R_2}. \tag{F.3}$$

So:

$$y_{inv} = -\frac{R_2}{R_1}a. \tag{F.4}$$

Differential Amplifier

In figure 3.4a we show the scheme of a Differential Amplifier. The current that flows through R_2 is the same, with a minus sign, as which passes on R_1. So, we can write:

$$\frac{a-u_-}{R_2}=\frac{u_- - y_{sub}}{R_1}. \tag{F.5}$$

The current that passes on R_3 is the same on R_4. Hence:

$$\frac{b-u_+}{R_3}=\frac{u_+}{R_4}, \tag{F.6}$$

and we get:

$$u_+=\frac{R_4}{R_3+R_4}b. \tag{F.7}$$

Now, because $u_+ = u_-$ and

$$y_{sub}=y_{sub}\left(a=0\right)+y_{sub}\left(b=0\right), \tag{F.8}$$

we can easily calculate y_{sub}. For $a=0$:

$$y_{sub}\left(a=0\right)=\frac{R_1+R_2}{R_2}\frac{R_4}{R_3+R_4}b. \tag{F.9}$$

And for $b=0$:

$$y_{sub}\left(b=0\right)=-\frac{R_1}{R_2}a. \tag{F.10}$$

Therefore:

$$y_{sub}=\frac{R_1+R_2}{R_2}\frac{R_4}{R_3+R_4}b-\frac{R_1}{R_2}a. \tag{F.11}$$

And if we decide to use all resistors with the same value, we get:

$$y_{sub}=b-a. \tag{F.12}$$

Inverting Differentiator

In figure 3.4b, and as we have been doing, we can already say:

$$C\frac{da}{dt}=-\frac{y_{dif}}{R}. \tag{F.13}$$

Such that:

$$y_{dif}=-RC\frac{da}{dt}. \tag{F.14}$$

Non-Inverting Weighted Summer

In figure 5.5 we have the scheme of a Summing Amplifier plus an Inverting Amplifier. The adding of an Inverting Amplifier is due to the sign resulted in the Summing Amplifier. In electronics, Summing Amplifier is the result x due to a and b and the resistors R_1, R_3 and R_4, and the result is $x = -(a+b)$ - considering all the resistors are the same. The procedure to get to this result is the same as we did to the Differential Amplifier, i.e., we say $a = 0$, and calculate the contribution of b, then we make $b = 0$ and calculate the contribution of a, and x will be the sum of the contributions. As a result of the minus sign, we add an Inverting Amplifier and the final result is $y = a+b$. Of course, with all resistors with different values, the result is:

$$y = \frac{R_2}{R_5}\left(\frac{R_3}{R_1}a + \frac{R_3}{R_2}b\right). \tag{F.15}$$

Non-Inverting Amplifier

In figure 5.6 we can see a Non-Inverting Amplifier, and with all the rules we already described, we can write:

$$\frac{a}{R_2} = \frac{y-a}{R_1}, \tag{F.16}$$

and we get:

$$y = \frac{R_1 + R_2}{R_2}a. \tag{F.17}$$

With $R_1 = R_2$:

$$y = 2a. \tag{F.18}$$

Appendix G

The Circuits

Experiments with electrical circuits are of the highest importance to analyse the results of a theory. Here is shown in figure G.1 a photograph of the electronic work, composed by the three oscillators and the two types of couplings. In figures G.2 and G.3, the backstage work done with the simulation program, before even stepping into a laboratory to build this model.

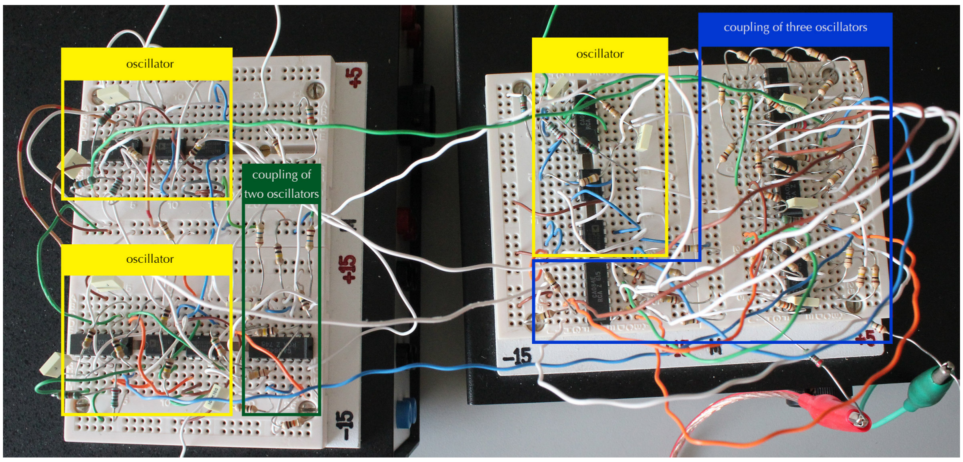


Figure G.1: Picture of the whole circuit of 2 and 3 coupled oscillators

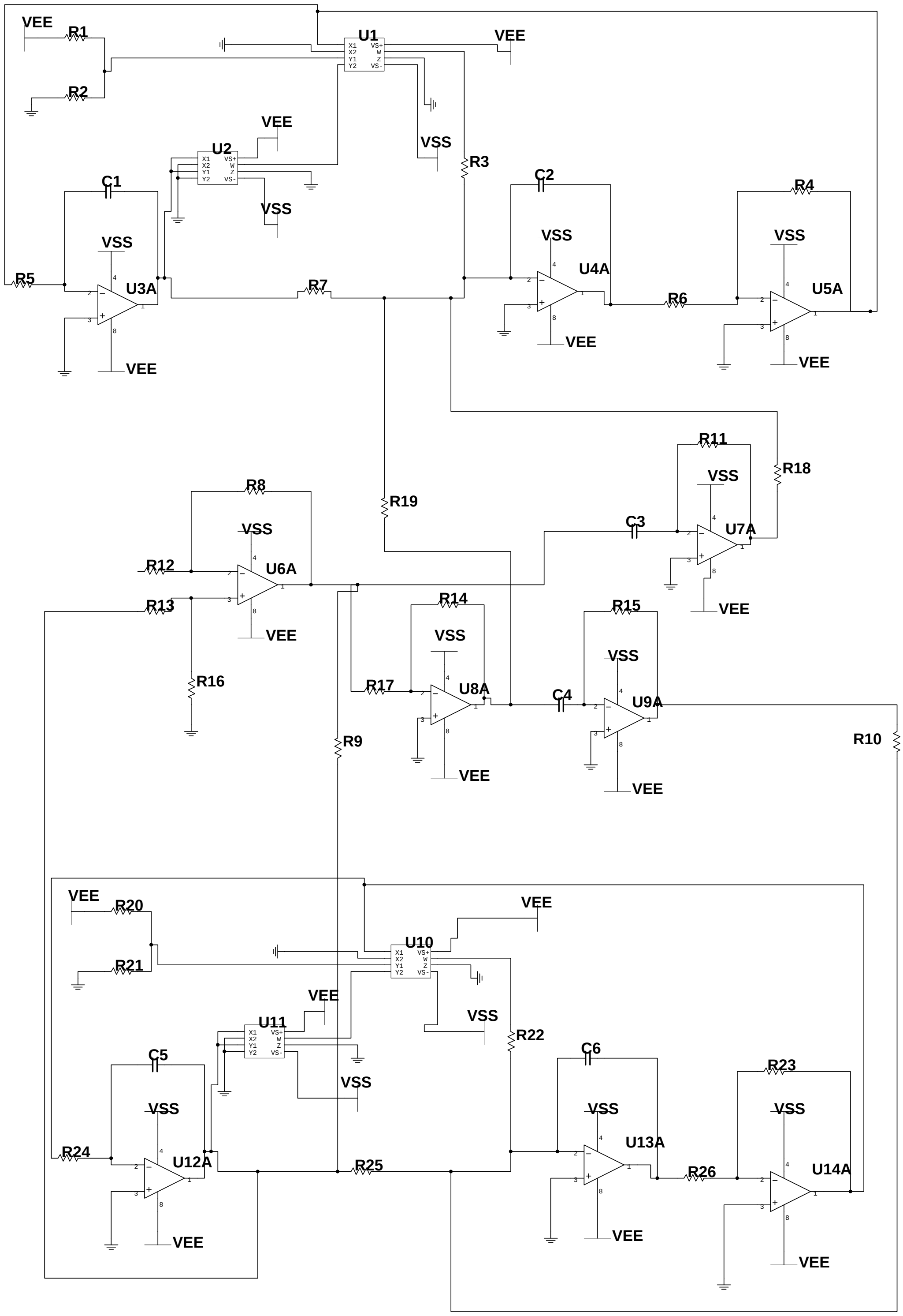


Figure G.2: Circuit scheme of two coupled oscillators.

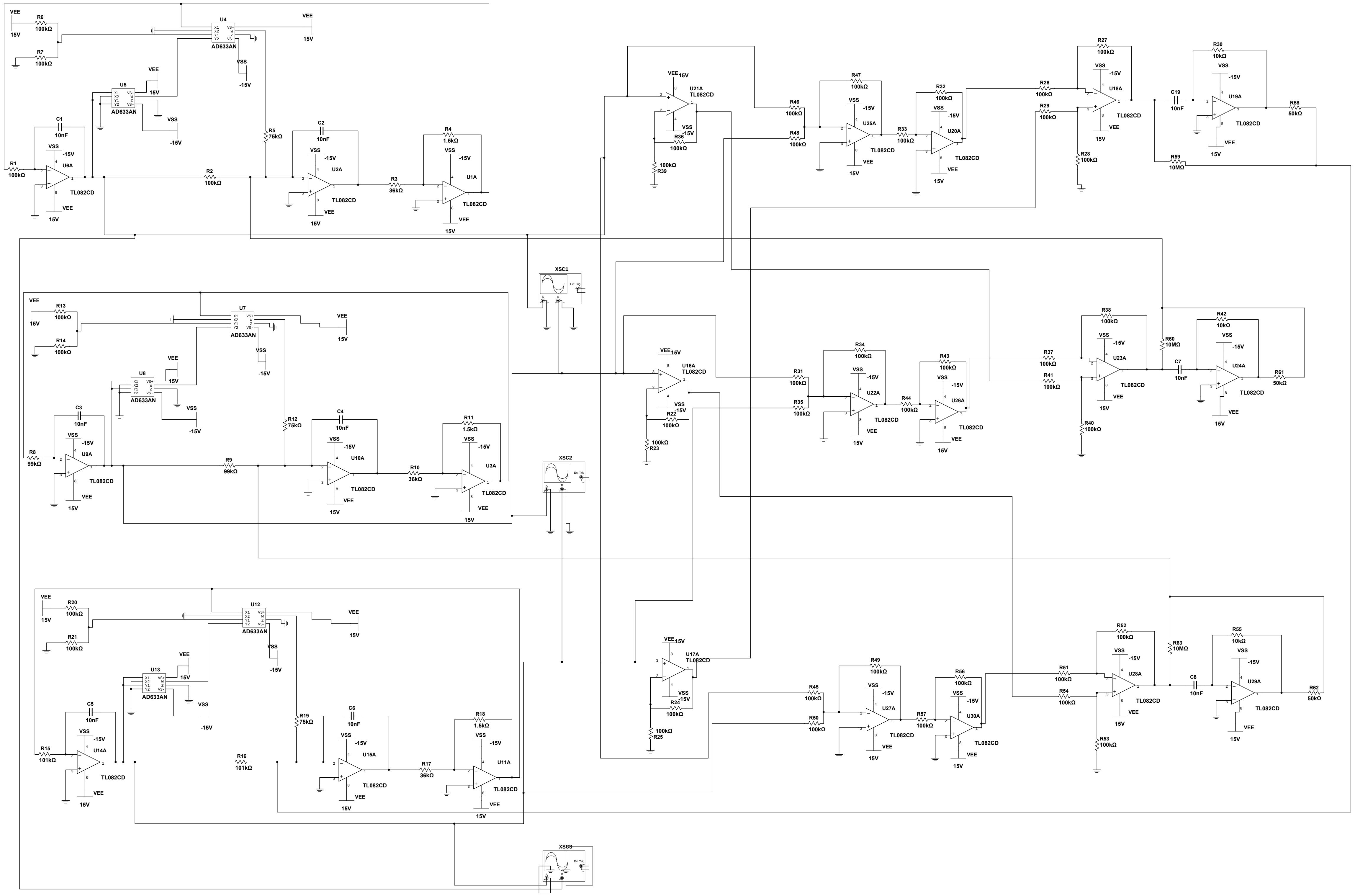


Figure G.3: Circuit scheme of three coupled oscillators.

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